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Nonabelian Bundle Gerbes, their Differential Geometry and Gauge Theory

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Abstract

Bundle gerbes are a higher version of line bundles, we present nonabelian bundle gerbes as a higher version of principal bundles. Connection, curving, curvature and gauge transformations are studied both in a global coordinate independent formalism and in local coordinates. These are the gauge fields needed for the construction of Yang-Mills theories with 2-form gauge potential.

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1. **Introduction**

Fibre bundles, besides being a central subject in geometry and topology, provide the mathematical framework for describing global aspects of Yang-Mills theories. Higher abelian gauge theories, i.e. gauge theories with abelian 2-form gauge potential appear naturally in string theory and field theory, and here too we have a corresponding mathematical structure, that of abelian gerbe (in algebraic geometry) and of abelian bundle gerbe (in differential geometry). Thus abelian bundle gerbes are a higher version of line bundles. Complex line bundles are geometric realizations of the integral 2nd cohomology classes $H^2(M,\mathbb{Z})$ on a manifold, i.e. the first Chern classes (whose de Rham representative is the field strength). Similarly, abelian (bundle) gerbes are the next level in realizing integral cohomology classes on a manifold, they are geometric realizations of the 3rd cohomology classes $H^3(M,\mathbb{Z})$. Thus the curvature 3-form of a 2-form gauge potential is the de Rham representative of a class in $H^3(M,\mathbb{Z})$. This class is called the Dixmier-Douady class [1],[2]; it topologically characterizes the abelian bundle gerbe in the same way that the first Chern class characterizes complex line bundles.

One way of thinking about abelian gerbes is in terms of their local transition functions [3],[4]. Local "transition functions" of an abelian gerbe are complex line bundles on double overlaps of open sets satisfying cocycle conditions for tensor products over quadruple overlaps of open sets. The nice notion of abelian bundle gerbe [5] is related to this picture. Abelian gerbes and bundle gerbes can be equipped with additional structures, that of connection 1-form and of curving (the 2-form gauge potential), and that of (bundle) gerbe modules (with or without connection and curving). Their holonomy can be introduced and studied [3],[4],[6],[7],[8],[9]. The equivalence class of an abelian gerbe with connection and curving is the Deligne class on the base manifold. The top part of the Deligne class is the class of the curvature, the Diximier-Douady class.

Abelian gerbes arise in a natural way in quantum field theory [10],[11],[12], where their appearance is due to the fact that one has to deal with abelian extensions of the group of gauge transformations; this is related to chiral anomalies. Gerbes and gerbe modules appear also very naturally in TQFT [13], in the WZW model [14] and in the description of D-brane anomalies in nontrivial background 3-form H-field (identified with the Diximier-Douday class) [15],[16],[17]. Coinciding (possibly infinitely many) D-branes are submanifolds "supporting" bundle gerbe modules [6] and can be classified by their (twisted) K-theory. The relation to the boundary conformal field theory description of D-branes is due to the identification of equivariant twisted K-theory with the Verlinde algebra [18],[19]. For the role of K-theory in D-brane physics see e.g. [20],[21],[22].

In this paper we study the nonabelian generalization of abelian bundle gerbes and their differential geometry, in other words we study higher Yang-Mills fields. Nonabelian gerbes arose in the context of nonabelian cohomology [23],[1] (see [24] for a concise introduction), see also ([25]). Their differential geometry –from the algebraic geometry point of view– is discussed thoroughly in the recent work of Breen and Messing [26] (and their combinatorics in [27]). Our study on the other hand is from the differential geometry viewpoint. We show that nonabelian bundle gerbes connections and curvings are very natural concepts also in classical differential geometry. We believe that it is primarily in this context that these structures can appear and can be recognized in physics. It is for example in this context that one would like to have a formulation of Yang-Mills theory with higher forms. These theories should be relevant in order to describe coinciding

NS5-branes with D2-branes ending on them. They should be also relevant in the study of M5-brane anomaly. We refer to [28], [29], [30] for some attempts in constructing higher gauge fields.

Abelian bundle gerbes are constructed using line bundles and their products. One can also study $U(1)$ bundle gerbes, here line bundles are replaced by their corresponding principal $U(1)$ bundles. In the study of nonabelian bundle gerbes it is more convenient to work with nonabelian principal bundles then with vector bundles. Actually principal bundles with additional structures are needed. We call these objects (principal) bibundles and D-H bundles (D and H being Lie groups). Bibundles are fibre bundles (with fiber H) which are at the same time left and right principal bundles (in a compatible way). They are the basic objects for constructing (principal) nonabelian bundle gerbes. The first part of this paper is therefore devoted to their description. In Section 2 we introduce bibundles, D-H bundles (i.e. principal D bundles with extra H structure) and study their products. In Section 3 we study the differential geometry of bibundles, in particular we define connections, covariant exterior derivatives and curvatures. These structures are generalizations of the corresponding structures on usual principal bundles. We thus describe them using a language very close to that of the classical reference books [31] or [32]. In particular a connection on a bibundle needs to satisfy a relaxed equivariance property, this is the price to be paid in order to incorporate nontrivially the additional bibundle structure. We are thus lead to introduce the notion of a 2-connection $(\boldsymbol{a}, A)$ on a bibundle. Products of bibundles with connections give a bibundle with connection only if the initial connections were compatible, we call this compatibility the summability conditions for 2-connections; a similar summability condition is established also for horizontal forms (e.g. 2-curvatures).

In Section 4, using the product between bibundles we finally introduce (principal) bundle gerbes. Here too we first describe their structure (including stable equivalence) and then only later in Section 7 we describe their differential geometry. We start with the proper generalization of abelian bundle gerbes in the sense of Murray [5], we then describe the relation to the Hitchin type presentation [3],[4], where similarly to the abelian case, nonabelian gerbes are described in terms of their "local transition functions" which are bibundles on double overlaps of open sets. The properties of the products of these bibundles over triple and quadruple overlaps define the gerbe and its nonabelian Čech 2-cocycle.

Section 5 is devoted to the example of the lifting bundle gerbe associated with the group extension $H \to E \to G$. In this case the bundle gerbe with structure group H appears as an obstruction to lift to E a G-principal bundle P.

Again by generalizing the abelian case, bundle gerbe modules are introduced in Section 6. Since we consider principal bibundles we obtain modules that are D-H bundles (compatible with the bundle gerbe structure). With each bundle gerbe there is canonically associated an $Aut(H)$-H bundle. In the lifting bundle gerbe example a module is given by the trivial E-H bundle.

In Section 7 we introduce the notion of bundle gerbe connection and prove that on a bundle gerbe a connection always exists. Bundle gerbe connections are then equivalently described as collections of local 2-connections on local bibundles (the "local transition functions of the bundle gerbe") satisfying a nonabelian cocycle condition on triple overlaps of open sets. Given a bundle gerbe connection we immediately have a connection on the canonical bundle gerbe module *can*. We describe also the case of a bundle gerbe

connection associated with an arbitrary bundle gerbe module. In particular we describe the bundle gerbe connection in the case of a lifting bundle gerbe.

Finally in Section 8 we introduce the nonabelian curving $\boldsymbol{b}$ (the 2-form gauge potential) and the corresponding nonabelian curvature 3-form $\boldsymbol{h}$. These forms are the nonabelian generalizations of the string theory B and H fields.

2. **Principal Bibundles and their Products**

Bibundles (bitorsors) where first studied by Grothendieck [33] and Giraud [1], their cohomology was studied in [34]. We here study these structures using the language of differential geometry.

Given two $U(1)$ principal bundles E, $\tilde{E}$, on the same base space M, one can consider the fiber product bundle $E\tilde{E}$, defined as the $U(1)$ principal bundle on M whose fibers are the product of the E and $\tilde{E}$, fibers. If we introduce a local description of E and $\tilde{E}$, with transition functions h^{ij} and $\tilde{h}^{ij}$ (relative to the covering $\{U^i\}$ of M), then $E\tilde{E}$ has transition functions $h^{ij}\tilde{h}^{ij}$.

In general, in order to multiply principal nonabelian bundles one needs extra structure. Let E and $\tilde{E}$ be H-principal bundles, we use the convention that H is acting on the bundles from the left. Then in order to define the H principal left bundle $E\tilde{E}$ we need also a right action of H on E. We thus arrive at the following

Definition 1. *An H principal bibundle E on the base space M is a bundle on M that is both a left H principal bundle and a right H principal bundle and where left and right H actions commute*

$$\forall\, h,k \in H\ ,\ \forall e \in E,\ \ (k\,e) \triangleleft h = k(e \triangleleft h)\ ; \tag{1}$$

we denote with $p\,: E \to M$ the projection to the base space.

Before introducing the product between principal bibundles we briefly study their structure. A morphism W between two principal bibundles E and $\tilde{E}$ is a morphism between the bundles E and $\tilde{E}$ compatible with both the left and the right action of H:

$$W(k\,e \triangleleft h) = k\,W(e)\tilde{\triangleleft}\,h\ , \tag{2}$$

here $\tilde{\triangleleft}$ is the right action of H on $\tilde{E}$. As for morphisms between principal bundles on the same base space M, we have that morphisms between principal bibundles on M are isomorphisms.

Trivial bibundles.
Since we consider only principal bibundles we will frequently write bibundle for principal bibundle. The product bundle $M \times H$ where left and right actions are the trivial ones on H [i.e. $k\,(x,h) \triangleleft h' = (x, khh')$] is a bibundle. We say that a bibundle T is trivial if T is isomorphic to $M \times H$.

Proposition 2. *We have that T is trivial as a bibundle iff it has a global central section, i.e. a global section $\boldsymbol{\sigma}$ that intertwines the left and the right action of H on T:*

$$\forall h \in H\ , \forall x \in M,\ \ h\,\boldsymbol{\sigma}(x) = \boldsymbol{\sigma}(x) \triangleleft h \tag{3}$$

Proof. Let $\boldsymbol{\sigma}$ be a global section of T, define $W_{\boldsymbol{\sigma}}\ :\ M \times H \to T$ as $W_{\boldsymbol{\sigma}}(x,h) = h\,\boldsymbol{\sigma}(x)$, then T and $M \times H$ are isomorphic as left principal bundles. The isomorphism $W_{\boldsymbol{\sigma}}$ is also a right principal bundles isomorphism iff (3) holds. $\square$

Note also that the section $\boldsymbol{\sigma}$ is unique if H has trivial centre. An example of nontrivial bibundle is given by the trivial left bundle $M \times H$ equipped with the nontrivial right action $(x, h) \triangleleft h' = (x, h\chi(h'))$ where χ is an outer automorphism of H. We thus see that bibundles are in general *not* locally trivial. Short exact sequences of groups provide examples of bibundles that are in general nontrivial as left bundles [cf. (112), (113)].

The φ map.
We now further characterize the relation between left and right actions. Given a bibundle E, the map $\boldsymbol{\varphi} : E \times H \to H$ defined by

$$\forall e \in E\,,\ \forall h \in H\,,\ \boldsymbol{\varphi}_e(h)\, e = e \triangleleft h \tag{4}$$

is well defined because the left action is free, and transitive on the fibers. For fixed $e \in E$ it is also one-to-one since the right action is transitive and left and right actions are free. Using the compatibility between left and right actions it is not difficult to show that $\boldsymbol{\varphi}$ is equivariant w.r.t. the left action and that for fixed $e \in E$ it is an automorphism of H:

$$\boldsymbol{\varphi}_{he}(h') = h\boldsymbol{\varphi}_e(h')h^{-1}\ , \tag{5}$$

$$\boldsymbol{\varphi}_e(hh') = \boldsymbol{\varphi}_e(h)\boldsymbol{\varphi}_e(h')\ , \tag{6}$$

we also have

$$\boldsymbol{\varphi}_{e\triangleleft h}(h') = \boldsymbol{\varphi}_e(hh'h^{-1})\ . \tag{7}$$

Vice versa given a left bundle E with an equivariant map $\boldsymbol{\varphi} : E \times H \to H$ that restricts to an H automorphism $\boldsymbol{\varphi}_e$, we have that E is a bibundle with right action defined by (4).

Using the $\boldsymbol{\varphi}$ map we have that a global section $\boldsymbol{\sigma}$ is a global central section (i.e. that a trivial left principal bundle is trivial as bibundle) iff [cf. (3)], $\forall x \in M$ and $\forall h \in H$,

$$\boldsymbol{\varphi}_{\boldsymbol{\sigma}(x)}(h) = h\ . \tag{8}$$

In particular, since $e \in E$ can be always written as $e = h'\boldsymbol{\sigma}$, we see that $\boldsymbol{\varphi}_e$ is always an *inner* automorphism,

$$\boldsymbol{\varphi}_e(h) = \boldsymbol{\varphi}_{h'\boldsymbol{\sigma}}(h) = Ad_{h'}(h)\ . \tag{9}$$

Vice versa, we have that

Proposition 3. *If H has trivial centre then an H bibundle E is trivial iff $\boldsymbol{\varphi}_e$ is an inner automorphism for all $e \in E$.*

Proof. Consider the local sections $\boldsymbol{t}^i : U^i \to E$, since H has trivial centre the map $k(\boldsymbol{t}) : U^i \to H$ is uniquely defined by $\boldsymbol{\varphi}_{\boldsymbol{t}^i}(h') = Ad_{k(\boldsymbol{t}^i)}h'$. From (5), $\boldsymbol{\varphi}_{h\boldsymbol{t}^i}(h') = Ad_h\, Ad_{k(\boldsymbol{t}^i)}h'$ and therefore the sections $k(\boldsymbol{t}^i)^{-1}\boldsymbol{t}^i$ are central because they satisfy $\boldsymbol{\varphi}_{k(\boldsymbol{t}^i)^{-1}\boldsymbol{t}^i}(h') = h'$. In the intersections $U^{ij} = U^i \cap U^j$ we have $\boldsymbol{t}^i = h^{ij}\boldsymbol{t}^j$ and therefore $k(\boldsymbol{t}^i)^{-1}\boldsymbol{t}^i = k(\boldsymbol{t}^j)^{-1}\boldsymbol{t}^j$. We can thus construct a global central section. □

Any principal bundle with H abelian is a principal bibundle in a trivial way, the map $\boldsymbol{\varphi}$ is given simply by $\boldsymbol{\varphi}_e(h) = h$.

Now let us recall that a global section $\boldsymbol{\sigma} : M \to E$ on a principal H-bundle $E \to M$ can be identified with an H-equivariant map $\overline{\sigma} : E \to H$. With our (left) conventions, $\forall E \in E$,

$$e = \overline{\sigma}(e)\boldsymbol{\sigma}(x)\ .$$

Notice, by the way, that if E is a trivial bibundle with a global section $\boldsymbol{\sigma}$, then $\overline{\sigma}$ is bi-equivariant, i.e.: $\overline{\sigma}(heh') = h\overline{\sigma}(e)h'$ iff σ is central. We apply this description of

a global section of a left principal bundle to the following situation. Consider an H-bibundle E. Let us form $Aut(H) \times_H E$ with the help of the canonical homomorphism $Ad : H \to Aut(H)$. Then it is straightforward to check that $\overline{\sigma} : [\eta, e] \mapsto \eta \circ \boldsymbol{\varphi}_e$ with $\eta \in Aut(H)$ is a global section of the left $Aut(H)$-bundle $Aut(H) \times_H E$. So $Aut(H) \times_H E$ is trivial as a left $Aut(H)$-bundle. On the other hand if E is a left principal H-bundle such that $Aut(H) \times_H E$ is a trivial left $Aut(H)$-bundle then it has a global section $\overline{\sigma} : Aut(H) \times_H E \to Aut(H)$ and the structure of an H-bibundle on E is given by $\boldsymbol{\varphi}_e \equiv \overline{\sigma}([id, e])$. We can thus characterize H-bibundles without mentioning their right H structure,

Proposition 4. *A left H-bundle E is an H-bibundle if and only if the (left) $Aut(H)$-bundle $Aut(H) \times_H E$ is trivial.*

Any trivial left H-bundle T can be given a trivial H-bibundle structure. We consider a trivialization of T i.e. an isomorphism $T \to M \times H$ and pull back the trivial right H-action on $M \times H$ to T. This just means that the global section of the left H-bundle T associated with the trivialization $T \to M \times H$, is by definition promoted to a global central section.

Product of bibundles.
In order to define the product bundle $E\tilde{E}$ we first consider the fiber product (Withney sum) bundle

$$E \oplus \tilde{E} \equiv \{(e, \tilde{e}) \mid p(e) = \tilde{p}(\tilde{e})\} \tag{10}$$

with projection $\rho : E \oplus \tilde{E} \to M$ given by $\rho(e, \tilde{e}) = p(e) = \tilde{p}(\tilde{e})$. We now can define the product bundle $E\tilde{E}$ with base space M via the equivalence relation

$$\forall h \in H \ (e, h\tilde{e}) \sim (e \triangleleft h \,, \tilde{e}) \tag{11}$$

we write $[e, \tilde{e}]$ for the equivalence class and

$$E\tilde{E} \equiv \ E \oplus_H \tilde{E} \equiv \{\, [e, \tilde{e}] \,\} \tag{12}$$

the projection $p\tilde{p} : E\tilde{E} \to M$ is given by $p\tilde{p}[e, \tilde{e}] = p(e) = \tilde{p}(\tilde{e})$. One can show that $E\tilde{E}$ is an H principal bundle; the action of H on $E\tilde{E}$ is inherited from that on E: $h[e, \tilde{e}] = [he, \tilde{e}]$. Concerning the product of sections we have that if $\boldsymbol{s} : U \to E$ is a section of E (with $U \subseteq M$), and $\tilde{\boldsymbol{s}} : U \to \tilde{E}$ is a section of $\tilde{E}$, then

$$\boldsymbol{s}\tilde{\boldsymbol{s}} \equiv [\, \boldsymbol{s}, \tilde{\boldsymbol{s}} \,] \ : \ U \to E\tilde{E} \tag{13}$$

is the corresponding section of $E\tilde{E}$.

When also $\tilde{E}$ is an H principal bibundle, with right action $\tilde{\triangleleft}$, then $E\tilde{E}$ is again an H principal bibundle with right action $\triangleleft\!\tilde{\triangleleft}$ given by

$$[e, \tilde{e}] \triangleleft\!\tilde{\triangleleft}\, h \ = \ [e, \tilde{e} \,\tilde{\triangleleft}\, h] \tag{14}$$

It is easy to prove that the product between H principal bibundles is associative.

Inverse bibundle.
The inverse bibundle E^{-1} of E has by definition the same total space and base space of E but the left action and the right actions $\triangleleft^{-1}$ are defined by

$$h\, e^{-1} = (e \triangleleft h^{-1})^{-1} \ , \ e^{-1} \triangleleft^{-1} h = (h^{-1} e)^{-1} \tag{15}$$

here e^{-1} and e are the same point of the total space, we write e^{-1} when the total space is endowed with the E^{-1} principal bibundle structure, we write e when the total space is endowed with the E principal bibundle structure. From the Definition (15) it follows that $he^{-1} = e^{-1} \lhd^{-1} \boldsymbol{\varphi}_e(h)$. Given the sections $\boldsymbol{t}^i : U^i \to E$ of E we canonically have the sections $\boldsymbol{t}^{i^{-1}} : U^i \to E^{-1}$ of E^{-1} (here again $\boldsymbol{t}^i(x)$ and $\boldsymbol{t}^{i^{-1}}(x)$ are the same point of the total space). The section $\boldsymbol{t}^{i^{-1}}\boldsymbol{t}^i$ of $E^{-1}E$ is central, i.e. it satisfies (3). We also have $\boldsymbol{t}^{i^{-1}}\boldsymbol{t}^i = \boldsymbol{t}^{j^{-1}}\boldsymbol{t}^j$ in U^{ij}; we can thus define a canonical (natural) global central section $\mathcal{I}$ of $E^{-1}E$, thus showing that $E^{-1}E$ is canonically trivial. Explicitly we have $\overline{I}[e'^{-1}, e] = h$ with $e' \lhd h = e$. Similarly for EE^{-1}. The space of isomorphism classes of H-bibundles on M [cf. (2)] can now be endowed with a group structure. The unit is the isomorphism class of the trivial product bundle $M \times H$. The inverse of the class represented by E is the class represented by E^{-1}.

Consider two isomorphic bibundles E and E' on M. The choice of a specific isomorphism between E and E' is equivalent to the choice of a global central section of the bibundle EE'^{-1}, i.e. a global section that satisfies (3). Indeed, let $\boldsymbol{f}$ be a global section of EE'^{-1}, given an element $e \in E$ with base point $x \in M$, there is a unique element $e'^{-1} \in E'^{-1}$ with base point $x \in M$ such that $[e, e'^{-1}] = \boldsymbol{f}(x)$. Then the isomorphism $E \sim E'$ is given by $e \mapsto e'$; it is trivially compatible with the right H-action, it is compatible with the left H-action because of the centrality of $\boldsymbol{f}$.

More generally let us consider two isomorphic left H-bundles $E \overset{W}{\sim} E'$ which are not necessarily bibundles. Let us write a generic element $(e, e') \in E \oplus E'$ in the form $(e, hW(e))$ with a properly chosen $h \in H$. We introduce an equivalence relation on $E \oplus E'$ by $(e, hW(e)) \sim (h'e, hW(h'e))$. Then $T = E \oplus E'/\sim$ is a trivial left H-bundle with global section $\bar{\sigma}([e, hW(e)]) = h^{-1}$ (the left H-action is inherited from E). Recalling the comments after Proposition 4, we equip T with trivial H-bibundle structure and global central section $\bar{\sigma}$. Next we consider the product TE' and observe that any element $[[e, e'_1], e''_2] \in TE'$ can be written as $[[\tilde{e}, W(\tilde{e})], W(\tilde{e})]$ with a unique $\tilde{e} \in E$. We thus have a canonical isomorphism between E and TE' and therefore we write $E = TE'$. Vice versa if T is a trivial bibundle with global central section $\bar{\sigma} : T \to H$ and E,E' are left H-bundles and $E = TE'$, i.e E is canonically isomorphic to TE', then we can consider the isomorphism $E \overset{W}{\sim} E'$ defined by $W([t, e']) = \bar{\sigma}(t)e'$ (here $[t, e']$ is thought as an element of E because of the identification $E = TE'$). It is then easy to see that the trivial bibundle with section given by this isomorphism W is canonically isomorphic to the initial bibundle T.

We thus conclude that the choice of an isomorphism between two left H-bundles E and E' is equivalent to the choice of a trivialization (the choice of a global central section) of the bibundle T, in formulae

$$E \overset{W}{\sim} E' \iff E = TE' \tag{16}$$

where T has a given global central section.

Local coordinates description.

We recall that an atlas of charts for an H principal left bundle E with base space M is given by a covering $\{U^i\}$ of M, together with sections $\boldsymbol{t}^i : U^i \to E$ (the sections $\boldsymbol{t}^i$ determine isomorphisms between the restrictions of E to U^i and the trivial bundles $U^i \times H$). The transition functions $h^{ij} : U^{ij} \to H$ are defined by $\boldsymbol{t}^i = h^{ij}\boldsymbol{t}^j$. They satisfy

on U^{ijk} the cocycle condition

$$h^{ij}h^{jk} = h^{ik}\,.$$

On U^{ij} we have $h^{ij} = {h^{ji}}^{-1}$. A section $\boldsymbol{s}\ :\ U \to E$ has local representatives $\{s^i\}$ where $s^i\ :\ U \cap U^i \to H$ and in U^{ij} we have

$$s^i h^{ij} = s^j\ . \tag{17}$$

If E is also a bibundle we set

$$\varphi^i \equiv \boldsymbol{\varphi}_{\boldsymbol{t}^i}\ :\ U^i \to Aut(H) \tag{18}$$

and we then have $\forall\ h \in H\ ,\ \varphi^i(h)h^{ij} = h^{ij}\varphi^j(h)\ ,$ i.e.

$$Ad_{h^{ij}} = \varphi^i \circ {\varphi^j}^{-1}\ . \tag{19}$$

We call the set $\{h^{ij}, \varphi^i\}$ of transition functions and φ^i maps satisfying (19) a set of local data of E. A different atlas of E, i.e. a different choice of sections $\boldsymbol{t}'^i = r^i\boldsymbol{t}^i$ where $r^i\ :\ U^i \to H$ (we can always refine the two atlases and thus choose a common covering $\{U^i\}$ of M), gives local data

$$h'^{ij} = r^i h^{ij} {r^j}^{-1}\quad , \tag{20}$$

$$\varphi'^i = Ad_{r^i} \circ \varphi^i\quad . \tag{21}$$

We thus define two sets of local data $\{h^{ij}, \varphi^i\}$ and $\{h^{ij}, \varphi^i\}$ to be equivalent if they are related by (20), (21).

One can reconstruct an H-bibundle E from a given set of local data $\{h^{ij}, \varphi^i\}$ relative to a covering $\{U^i\}$ of M. For short we write $E = \{h^{ij}, \varphi^i\}$. The total space of this bundle is the set of triples (x, h, i) where $x \in U^i$, $h \in H$, modulo the equivalence relation $(x, h, i) \sim (x', h', j)$ iff $x = x'$ and $hh^{ij} = h'$. We denote the equivalence class by $[x, h, i]$. The left H action is $h'[x, h, i] = [x, h'h, i]$. The right action, given by $[x, h, i] \triangleleft h' = [x, h\varphi^i(h'), i]$ is well defined because of (19). The h^{ij}'s are transition functions of the atlas given by the sections $\boldsymbol{t}^i : U^i \to E$, $\boldsymbol{t}^i(x) = [x, 1, i]$, and we have $\boldsymbol{\varphi}_{\boldsymbol{t}^i} = \varphi^i$. It is now not difficult to prove that equivalence classes of local data are in one-to-one correspondence with isomorphism classes of bibundles. [Hint: ${\boldsymbol{t}'^i}^{-1}(r^i\boldsymbol{t}^i)$ is central and i independent].

Given two H bibundles $E = \{h^{ij}, \varphi^i\}$ and $\tilde{E} = \{\tilde{h}^{ij}, \tilde{\varphi}^i\}$ on the same base space M, the product bundle $E\tilde{E}$ has transition functions and left H-actions given by (we can always choose a covering $\{U^i\}$ of M common to E and $\tilde{E}$)

$$E\tilde{E} = \{h^{ij}\varphi^j(\tilde{h}^{ij})\,, \varphi^i \circ \tilde{\varphi}^i\} \tag{22}$$

If $\tilde{E}$ is not a bibundle the product $E\tilde{E}$ is still a well defined bundle with transition functions $h^{ij}\varphi^j(\widetilde{h^{ij}})$. Associativity of the product (22) is easily verified. One also shows that if $s^i\,, \tilde{s}^i\ :\ U \cap U^i \to H$ are local representatives for the sections $\boldsymbol{s}\ :\ U \to E$ and $\tilde{\boldsymbol{s}}\ :\ U \to \tilde{E}$ then the local representative for the product section $\boldsymbol{s}\tilde{\boldsymbol{s}}\ :\ U \to E\tilde{E}$ is given by

$$s^i\varphi^i(\tilde{s}^i)\ . \tag{23}$$

The inverse bundle of $E = \{h^{ij}, \varphi^i\}$ is

$$E^{-1} = \{{\varphi^j}^{-1}({h^{ij}}^{-1})\,, {\varphi^i}^{-1}\} \tag{24}$$

(we also have ${\varphi^j}^{-1}(h^{ij})^{-1} = {\varphi^i}^{-1}({h^{ij}}^{-1})\,)$. If $\boldsymbol{s}\ :\ U \to E$ is a section of E with representatives $\{s^i\}$ then $\boldsymbol{s}^{-1}\ :\ U \to E^{-1}$, has representatives $\{{\varphi^i}^{-1}({s^i}^{-1})\}$.

A trivial bundle T with global central section $\boldsymbol{t}$, in an atlas of charts subordinate to a cover U^i of the base space M, reads

$$T = \{f^i {f^j}^{-1},\, Ad_{f^i}\}\ , \tag{25}$$

where the section $\boldsymbol{t} \equiv \boldsymbol{f}^{-1}$ has local representatives $\{{f^i}^{-1}\}$. For future reference notice that $T^{-1} = \{{f^i}^{-1} f^j,\, Ad_{{f^i}^{-1}}\}$ has global central section $\boldsymbol{f} = \{f^i\}$, and that $ET^{-1}E^{-1}$ is trivial,

$$ET^{-1}E^{-1} = \{\varphi^i({f^i}^{-1})\varphi^j(f^j),\, Ad_{\varphi^i({f^i}^{-1})}\}\ . \tag{26}$$

We denote by $\boldsymbol{\varphi}(\boldsymbol{f})$ the global central section $\{\varphi^i(f^i)\}$ of $ET^{-1}E^{-1}$. Given an *arbitrary* section $\boldsymbol{s}\ :\ U \to E$, we have, in U

$$\boldsymbol{\varphi}(\boldsymbol{f}) = \boldsymbol{s}\boldsymbol{f}\boldsymbol{s}^{-1} \tag{27}$$

Proof : $\boldsymbol{f}\boldsymbol{s}^{-1} = \{f^i Ad_{{f^i}^{-1}}({\varphi^i}^{-1}({s^i}^{-1}))\} = \{{\varphi^i}^{-1}({s^i}^{-1})f^i\}$ and therefore $\boldsymbol{s}\boldsymbol{f}\boldsymbol{s}^{-1} = \{\varphi^i(f^i)\} = \boldsymbol{\varphi}(\boldsymbol{f})$. Property (27) is actually the defining property of $\boldsymbol{\varphi}(\boldsymbol{f})$. Without using an atlas of charts, we define the global section $\boldsymbol{\varphi}(\boldsymbol{f})$ of $ET^{-1}E^{-1}$ to be that section that locally satisfies (27). The definition is well given because centrality of $\boldsymbol{f}$ implies that $\boldsymbol{\varphi}(\boldsymbol{f})$ is independent from $\boldsymbol{s}$. Centrality of the global section $\boldsymbol{f}$ also implies that $\boldsymbol{\varphi}(\boldsymbol{f})$ is a global central section. If $\overline{\sigma}$ is the global central section of T, the corresponding global section $\overline{\sigma}'$of $ET^{-1}E^{-1}$ is $\overline{\sigma}'[e,t,e'^{-1}] = \varphi_e(\overline{\sigma}(t))h$ with $e = he'$.

The pull-back of a bi-principal bundle is again a bi-principal bundle. It is also easy to verify that the pull-back commutes with the product.

***D*-*H* bundles.**

We can generalize the notion of a bibundle by introducing the concept of a crossed module.

We say that H is a crossed D-module [35] if there is a group homomorphism $\alpha : H \to D$ and an action of D on H denoted as $(d,h) \mapsto {}^d h$ such that

$$\forall h,\, h' \in H\ ,\ \ {}^{\alpha(h)}h' = hh'h^{-1} \tag{28}$$

and forall $h \in H$, $d \in D$,

$$\alpha({}^d h) = d\alpha(h)d^{-1} \tag{29}$$

holds true.

Notice in particular that $\alpha(H)$ is normal in D. The canonical homomorphism $Ad : H \to Aut(H)$ and the canonical action of $Aut(H)$ on H define on H the structure of a crossed $Aut(H)$-module. Given a D-bundle Q we can use the homomorphism $t : D \to Aut(H)$, $t \circ \alpha = Ad$ to form $Aut(H) \times_D Q$.

Definition 5. *Consider a left D-bundle Q on M such that the $Aut(H)$-bundle $Aut(H) \times_D Q$ is trivial. Let σ be a global section of $Aut(H) \times_D Q$. We call the couple (Q,σ) a D-H bundle.*

Notice that if $\overline{\sigma} : Aut(H) \times_D Q \ni [\eta, q] \mapsto \overline{\sigma}([\eta,q]) \in Aut(H)$ is a global section of $Aut(H) \times_D Q$ then

i) the automorphism $\boldsymbol{\psi}_q \in Aut(H)$ defined by

$$\boldsymbol{\psi}_q \equiv \overline{\sigma}([id, q]) \tag{30}$$

is D-equivariant,

$$\boldsymbol{\psi}_{dq}(h) = {}^d\boldsymbol{\psi}_q(h) \tag{31}$$

ii) the homomorphism $\boldsymbol{\xi}_q : H \to D$ defined by

$$\boldsymbol{\xi}_q(h) \equiv \alpha \circ \boldsymbol{\psi}_q(h) \tag{32}$$

gives a fiber preserving action $q \triangleleft h \equiv \boldsymbol{\xi}_q(h)q$ of H on the right, commuting with the left D-action. i.e.

$$\forall\ h \in H, d \in D, q \in Q, \qquad (d\,q) \triangleleft h = d(q \triangleleft h)\ . \tag{33}$$

Vice versa we easily have

Proposition 6. *Let H be a crossed D-module. If Q is a left D bundle admitting a right fiber preserving H action commuting with the left D action, and the homomorphism $\boldsymbol{\xi}_q : H \to D$, defined by $q \triangleleft h = \boldsymbol{\xi}_q(h)q$ is of the form (32) with a D-equivariant $\boldsymbol{\psi}_q \in Aut(H)$ [cf. (31)], then Q is a D-H bundle.*

There is an obvious notion of an isomorphism between two D-H bundles (Q, σ) and $(\tilde{Q}, \tilde{\sigma})$; it is an isomorphism between D-bundles Q and $\tilde{Q}$ intertwining between σ and $\tilde{\sigma}$. In the following we denote a D-H bundle (Q, σ) simply as Q without spelling out explicitly the choice of a global section σ of $Aut(H) \times_D Q$. As in the previous section out of a given isomorphism we can construct a trivial D-bibundle Z with a global central section $\boldsymbol{z}^{-1}$ such that $\tilde{Q}$ and ZQ are canonically identified and we again write this as $\tilde{Q} = ZQ$. The $\boldsymbol{\psi}$ map of Z is given by $Ad_{\overline{z}^{-1}}$.

Note that the product of a trivial D-bibundle Z and a D-H bundle Q is well-defined and gives again a D-H bundle.

The trivial bundle $M \times D \to M$, with right H-action given by $(x, d) \triangleleft h = (x, d\alpha(h))$, is a D-H bundle, we have $\boldsymbol{\psi}_{(x,d)}(h) = {}^d h$. A D-H bundle Q is trivial if it is isomorphic to $M \times D$. Similarly to the case of a bibundle we have that a D-H bundle is trivial iff it has a global section $\boldsymbol{\sigma}$ which is central with respect to the left and the right actions of H on Q,

$$\boldsymbol{\sigma}(x) \triangleleft h = \alpha(h)\boldsymbol{\sigma}(x)\ . \tag{34}$$

The corresponding map $\overline{\sigma} : Q \to D$ is then bi-equivariant

$$\overline{\sigma}(dq \triangleleft h) = d\,\overline{\sigma}(q)\alpha(h). \tag{35}$$

The pull-back of a D-H bundle is again a D-H bundle.

The trivial bundle $Aut(H) \times_H E$ (cf. Proposition 4) is an $Aut(H)$-H bundle. *Proof.* The left $Aut(H)$ and the right H actions commute, and they are related by $[\eta, e]h = Ad_{\eta(\boldsymbol{\varphi}_e(h))}[\eta, e]$; we thus have $\boldsymbol{\psi}_{[\eta,e]} = \eta \circ \boldsymbol{\varphi}_e$, which structures $Aut(H) \times_H E$ into an $Aut(H)$-H bundle. Moreover $\overline{\sigma}([\eta, e]) = \eta \circ \boldsymbol{\varphi}_e$ is bi-equivariant, hence $Aut(H) \times_H E$ is isomorphic to $M \times Aut(H)$ as an $Aut(H)$-H bundle.

More generally, we can use the left H-action on D given by the homomorphism $\alpha : H \to D$ to associate to a bibundle E the bundle $D \times_H E$. The H-automorphism $\boldsymbol{\psi}_{[d,e]}$ defined by $\boldsymbol{\psi}_{[d,e]} = {}^d\boldsymbol{\varphi}_e(h)$ endows $D \times_H E$ with a D-H bundle structure.

There is the following canonical construction associated with a D-H module. We use the D-action on H to form the associated bundle $H \times_D Q$. Using the equivariance property (31) of $\boldsymbol{\psi}_q$ we easily get the following proposition.

Proposition 7. *The associated bundle $H \times_D Q$ is a trivial H-bibundle with actions $h'[h, q] = [\boldsymbol{\psi}_q(h')h, q]$ and $[h, q] \triangleleft h' = [h\boldsymbol{\psi}_q(h'), q]$, and with global central section given by $\bar{\sigma}([h, q]) = \boldsymbol{\psi}_q^{-1}(h)$.*

The local coordinate description of a D-H bundle Q is similar to that of a bibundle. We thus omit the details. We denote by d^{ij} the transition functions of the left principal D-bundle Q. Instead of local maps (18) we now have local maps $\psi^i : U^i \to Aut(H)$, such that (compare to (19))

$$^{d_{ij}}h = \psi^i \circ \psi^{j^{-1}}(h). \tag{36}$$

The product QE of a D-H bundle Q with a H-bibundle E can be defined as in (11), (12). The result is again a D-H bundle. If Q is locally given by $\{d^{ij}, \psi^i\}$ and H is locally given by $\{h^{ij}, \varphi^i\}$ then QE is locally given by $\{d^{ij}\xi^j(h^{ij}), \psi^i \circ \varphi^i\}$. Moreover if $Z = \{z^i z^{j^{-1}}, Ad_{z^i}\}$ is a trivial D-bibundle with section $\boldsymbol{z}^{-1} = \{z^{i^{-1}}\}$, then the well-defined D-H bundle ZQ is locally given by $\{z^i d^{ij} z^{j^{-1}}, {}^{z^i} \circ \psi^i\}$. He have the following associativity property

$$(ZQ)E = Z(QE) \ , \tag{37}$$

and the above products commute with pull-backs.

Given a D-H bundle Q and a trivial H-bibundle T with section $\boldsymbol{f}^{-1}$ there exists a unique trivial D-bibundle $\boldsymbol{\xi}(T)$ with section $\boldsymbol{\xi}(\boldsymbol{f}^{-1})$ such that

$$QT = \boldsymbol{\xi}(T)Q \ , \tag{38}$$

i.e. such that for any local section $\boldsymbol{s}$ of Q one has $\boldsymbol{s}\boldsymbol{f}^{-1} = \boldsymbol{\xi}(\boldsymbol{f}^{-1})\boldsymbol{s}$. The notations $\boldsymbol{\xi}(T)$, $\boldsymbol{\xi}(\boldsymbol{f}^{-1})$ are inferred from the local expressions of these formulae. Indeed, if locally $T = \{f^i f^{j^{-1}}, Ad_{f^i}\}$ and $\boldsymbol{f} = \{f^i\}$, then $\boldsymbol{\xi}(T) = \{\xi^i(f^i)\xi^j(f^j)^{-1}, Ad_{\xi^i(f^i)}\}$ and $\boldsymbol{\xi}(\boldsymbol{f}) = \{\xi^i(f^i)\}$.

Finally, as was the case for bibundles, we can reconstruct a D-H bundle Q from a given set of local data $\{d^{ij}, \psi^i\}$ relative to a covering $\{U^i\}$ of M. Equivalence of local data for D-H bundles is defined in such a way that isomorphic (equivalent) D-H bundles have equivalent local data, and vice versa.

3. **Connection and Curvature on Principal Bibundles**

Since a bibundle E on M is a bundle on M that is both a left principal H-bundle and a right principal H-bundle, one could then define a connection on a bibundle to be a one-form $\boldsymbol{a}$ on E that is both a left and a right principal H-bundle connection. This definition [more precisely the requirement $\mathcal{A}^r = 0$ in (49)] preserves the left-right symmetry property of the bibundle structure, but it turns out to be too restrictive, indeed not always a bibundle can be endowed with such a connection, and furthermore the corresponding curvature is valued in the center of H. If we insist in preserving the left-right symmetry structure we are thus led to generalize (relax) the notion of connection. In this section we will see that a connection on a bibundle is a couple $(\boldsymbol{a}, A)$ where $\boldsymbol{a}$ is a one-form on E with values in Lie(H) while A is a Lie($Aut(H)$) valued one-form on M. In particular we see that if $A = 0$ then $\boldsymbol{a}$ is a left connection on E where E is considered just as a left principal bundle. We recall that a connection $\boldsymbol{a}$ on a left principal bundle E satisfies [31]

i) the pull-back of $\boldsymbol{a}$ on the fibers of E is the right invariant Maurer-Cartan one-form. Explicitly, let $e \in E$, let $g(t)$ be a curve from some open interval $(-\varepsilon, \varepsilon)$ of the real line into the group H with $g(0) = 1_H$, and let $[g(t)]$ denote the corresponding tangent vector in 1_H and $[g(t)e]$ the vertical vector based in $e \in E$. Then

$$\boldsymbol{a}[g(t)e] = -[g(t)] \ . \tag{39}$$

Equivalently $\boldsymbol{a}[g(t)e] = \zeta_{[g(t)]}$ where $\zeta_{[g(t)]}$ is the right-invariant vector field associated with $[g(t)] \in \mathrm{Lie}(H)$, i.e. $\zeta_{[g(t)]}|_h = -[g(t)h]$.

ii) under the left H-action we have the equivariance property

$$l^{h^*}\boldsymbol{a} = Ad_h\boldsymbol{a} \tag{40}$$

where l^h denotes left multiplication by $h \in H$.

Now property *i)* is compatible with the bibundle structure on E in the following sense, if $\boldsymbol{a}$ satisfies *i)* then $-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})$ pulled back on the fibers is the left invariant Maurer-Cartan one-form

$$-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})[eg(t)] = [g(t)] \ , \tag{41}$$

here with abuse of notation we use the same symbol $\boldsymbol{\varphi}^{-1}$ for the map $\boldsymbol{\varphi}^{-1} : E \times H \to H$ and its differential map $\boldsymbol{\varphi}_*^{-1} : E \times \mathrm{Lie}(H) \to \mathrm{Lie}(H)$. Property (41) is equivalent to $\boldsymbol{a}[g(t)e] = \xi_{[g(t)]}$ where $\xi_{[g(t)]}$ is the left-invariant vectorfield associated with $[g(t)] \in \mathrm{Lie}(H)$, i.e. $\xi_{[g(t)]}|_h = [hg(t)]$. Property (41) is easily proven,

$$-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})[eg(t)] = -\boldsymbol{\varphi}_e^{-1}(\boldsymbol{a}[\boldsymbol{\varphi}_e(g(t))e]) = \boldsymbol{\varphi}_e^{-1}[\boldsymbol{\varphi}_e(g(t))] = [g(t)] \ .$$

Similarly, on the vertical vectors $v_{_V}$ of E we have $\left(r^{h^*}\boldsymbol{a} - \boldsymbol{a}\right)(v_{_V}) = 0$, $\left(l^{h^*}\varphi^{-1}(\boldsymbol{a}) - \varphi^{-1}(\boldsymbol{a})\right)(v_{_V}) = 0$ and

$$\left(l^{h^*}\boldsymbol{a} - Ad_h\boldsymbol{a}\right)(v_{_V}) = 0 \ , \tag{42}$$

$$\left(r^{h^*}\varphi^{-1}(\boldsymbol{a}) - Ad_{h^{-1}}\varphi^{-1}(\boldsymbol{a})\right)(v_{_V}) = 0 \ . \tag{43}$$

On the other hand property *ii)* is not compatible with the bibundle structure, indeed if $\boldsymbol{a}$ satisfies (40) then it can be shown (see later) that $-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})$ satisfies

$$r^{h^*}\boldsymbol{\varphi}^{-1}(\boldsymbol{a}) = Ad_{h^{-1}}\boldsymbol{\varphi}^{-1}(\boldsymbol{a}) - p^*T'(h^{-1}) \tag{44}$$

where $T'(h)$ is a given one-form on the base space M, and $p : E \to M$. In order to preserve the left-right symmetry structure we are thus led to generalize (relax) the equivariance property *ii)* of a connection. Accordingly with (42) and (44) we thus require

$$l^{h^*}\boldsymbol{a} = Ad_h\boldsymbol{a} + p^*T(h) \tag{45}$$

where $T(h)$ is a one-form on M. From (45) it follows

$$T(hk) = T(h) + Ad_hT(k) \ , \tag{46}$$

i.e., T is a 1-cocycle in the group cohomology of H with values in $\mathrm{Lie}(H) \otimes \Omega^1(M)$. Of course if T is a coboundary, i.e. $T(h) = h\boldsymbol{\chi}h^{-1} - \boldsymbol{\chi}$ with $\boldsymbol{\chi} \in \mathrm{Lie}(H) \otimes \Omega^1(M)$, then $\boldsymbol{a} + \boldsymbol{\chi}$ is a connection. We thus see that eq. (45) is a nontrivial generalization of the equivariance property only if the cohomology class of T is nontrivial.

Given an element $X \in \mathrm{Lie}(Aut(H))$, we can construct a corresponding 1-cocycle T_X in the following way,

$$T_X(h) \equiv [he^{tX}(h^{-1})] \ ,$$

where $[he^{tX}(h^{-1})]$ is the tangent vector to the curve $he^{tX}(h^{-1})$ at the point 1_H; if H is normal in $Aut(H)$ then $e^{tX}(h^{-1}) = e^{tX}h^{-1}e^{-tX}$ and we simply have $T_X(h) = hXh^{-1} - X$.

Given a Lie$(Aut(H))$-valued one-form A on M, we write $A = A^\rho X^\rho$ where $\{X^\rho\}$ is a basis of Lie$(Aut(H))$. We then define T_A as

$$T_A \equiv A^\rho T_{X^\rho} \ . \tag{47}$$

Obviously, $p^*T_A = T_{p^*A}$. Following these considerations we define

Definition 8. *A 2-connection on E is a couple $(\boldsymbol{a}, A)$ where:*
i) $\boldsymbol{a}$ is a Lie(H) *valued one-form on E such that its pull-back on the fibers of E is the right invariant Maurer-Cartan one-form, i.e. $\boldsymbol{a}$ satisfies (39),*
ii) A is a Lie$(Aut(H))$ *valued one-form on M,*
iii) the couple $(\boldsymbol{a}, A)$ satisfies

$$l^{h^*}\boldsymbol{a} = Ad_h\boldsymbol{a} + p^*T_A(h) \ . \tag{48}$$

This definition seems to break the left-right bibundle symmetry since, for example, only the left H action has been used. This is indeed *not* the case

Theorem 9. *If $(\boldsymbol{a}, A)$ is a 2-connection on E then $(\boldsymbol{a}^r, A^r)$, where $\boldsymbol{a}^r \equiv -\boldsymbol{\varphi}^{-1}(\boldsymbol{a})$, satisfies (39) and (48) with the left H action replaced by the right H action (and right-invariant vectorfields replaced by left-invariant vectorfields), i.e. it satisfies (41) and*

$$r^{h^*}\boldsymbol{a}^r = Ad_{h^{-1}}\boldsymbol{a}^r + p^*T_{A^r}(h^{-1}) \ , \tag{49}$$

here A^r is the one-form on M uniquely defined by the property

$$p^*A^r = \boldsymbol{\varphi}^{-1}(p^*A + ad_{\boldsymbol{a}})\boldsymbol{\varphi} + \boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi} \ . \tag{50}$$

Proof. First we observe that from (39) and (48) we have

$$l^{h'^*}\boldsymbol{a} = Ad_{h'}\boldsymbol{a} + p^*T_A(h') + h'dh'^{-1} \tag{51}$$

where now $h' = h'(e)$, i.e. h' is an H-valued function on the total space E. Setting $h' = \boldsymbol{\varphi}(h)$, with $h \in H$ we have

$$\begin{aligned} r^{h^*}\boldsymbol{a} = l^{\boldsymbol{\varphi}(h)^*}\boldsymbol{a} &= Ad_{\boldsymbol{\varphi}(h)}\boldsymbol{a} + p^*T_A(\boldsymbol{\varphi}(h)) + \boldsymbol{\varphi}(h)d\boldsymbol{\varphi}(h^{-1}) \\ &= \boldsymbol{a} + \boldsymbol{\varphi}(T_{\mathcal{A}^r}(h)) \end{aligned} \tag{52}$$

in equality (52) we have defined

$$\mathcal{A}^r \equiv \boldsymbol{\varphi}^{-1}(p^*A + ad_{\boldsymbol{a}})\boldsymbol{\varphi} + \boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi} \, . \tag{53}$$

Equality (52) holds because of the following properties of T,

$$T_{\boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}}(h) = \boldsymbol{\varphi}^{-1}\left(\boldsymbol{\varphi}(h)d\boldsymbol{\varphi}(h^{-1})\right) \ , \tag{54}$$

$$T_{\boldsymbol{\varphi}^{-1}p^*A\boldsymbol{\varphi}}(h) = \boldsymbol{\varphi}^{-1}\left(T_{p^*A}(\boldsymbol{\varphi}(h))\right) \ , \tag{55}$$

$$T_{ad_{\boldsymbol{a}}}(h) = Ad_h\boldsymbol{a} - \boldsymbol{a} \ . \tag{56}$$

From (52), applying $\boldsymbol{\varphi}^{-1}$ and then using (7) one obtains

$$r^{h^*}\boldsymbol{a}^r = Ad_{h^{-1}}\boldsymbol{a}^r + T_{\mathcal{A}^r}(h^{-1}) \ . \tag{57}$$

Finally, comparing (43) with (57) we deduce that for all $h \in H$, $T_{\mathcal{A}^r}(h)(v_{_V}) = 0$, and this relation is equivalent to $\mathcal{A}^r(v_{_V}) = 0$. In order to prove that $\mathcal{A}^r = p^*A^r$ where A^r is a one-form on M, we then just need to show that $\mathcal{A}^r$ is invariant under the H action,

$l^{h^*}\mathcal{A}^r = \mathcal{A}^r$. This is indeed the case because $l^{h^*}(\boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}) = \boldsymbol{\varphi}^{-1}Ad_{h^{-1}}dAd_h\boldsymbol{\varphi} = \boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}$, and because

$$\begin{aligned} l^{h^*}\left(\boldsymbol{\varphi}^{-1}(p^*A + ad_{\boldsymbol{a}})\boldsymbol{\varphi}\right) &= \boldsymbol{\varphi}^{-1}Ad_{h^{-1}}(p^*A + l^{h^*}ad_{\boldsymbol{a}})Ad_h\boldsymbol{\varphi} \\ &= \boldsymbol{\varphi}^{-1}\left(Ad_{h^{-1}}p^*AAd_h + ad_{\boldsymbol{a}} + ad_{Ad_{h^{-1}}T_{p^*A}(h)}\right)\boldsymbol{\varphi} \\ &= \boldsymbol{\varphi}^{-1}(p^*A + ad_{\boldsymbol{a}})\boldsymbol{\varphi}\ . \end{aligned}$$

□

Notice that if $(\boldsymbol{a}, A)$ and $(\boldsymbol{a'}, A')$ are 2-connections on E then so is the affine sum

$$\left(\, p^*(\lambda)\boldsymbol{a} + (1 - p^*\lambda)\boldsymbol{a'}\ ,\ \lambda A + (1-\lambda)A'\,\right) \tag{58}$$

for any (smooth) function λ on M.

As in the case of principal bundles we define a vector $v \in T_eE$ to be horizontal if $\boldsymbol{a}(v) = 0$. The tangent space T_eE is then decomposed in the direct sum of its horizontal and vertical subspaces; for all $v \in T_eE$, we write $v = \mathrm{H}v + \mathrm{V}v$, where $Vv = [e^{-t\boldsymbol{a}(v)}e]$. The space of horizontal vectors is however not invariant under the usual left H-action, indeed

$$\boldsymbol{a}(l^h{}_*(\mathrm{H}v)) = T_A(h)(v)\ ,$$

in this formula, as well as in the sequel, with abuse of notation T_A stands for T_{p^*A}.

Remark 10. It is possible to construct a new left H-action $\mathcal{L}_*$ on T_*E, that is compatible with the direct sum decomposition $T_*E = \mathrm{H}T_*E + \mathrm{V}T_*E$. We first define, for all $h \in H$,

$$\begin{aligned} L^h_A\ :\ T_*E &\ \rightarrow\ \mathrm{V}T_*E\ , \\ T_eE \ni v &\ \mapsto\ [e^{tT_A(h)(v)}he] \in \mathrm{V}T_{he}E\ , \end{aligned} \tag{59}$$

and notice that L^h_A on vertical vectors is zero, therefore $L^h_A \circ L^h_A = 0$. We then consider the tangent space map,

$$\mathcal{L}^h_* \equiv l^h_* + L^h_A\ . \tag{60}$$

It is easy to see that $\mathcal{L}^{hk}_* = \mathcal{L}^h_* \circ \mathcal{L}^k_*$ and therefore that $\mathcal{L}_*$ defines an action of H on T_*H. We also have

$$\mathcal{L}^{h^*}\boldsymbol{a} = Ad_h\boldsymbol{a}\ . \tag{61}$$

Finally the action $\mathcal{L}^h_*$ preserves the horizontal and vertical decomposition $T_*E = \mathrm{H}T_*E + \mathrm{V}T_*E$, indeed

$$\mathrm{H}\mathcal{L}^h_*v = \mathcal{L}^h_*\mathrm{H}v \quad , \quad \mathrm{V}\mathcal{L}^h_*v = \mathcal{L}^h_*\mathrm{V}v \tag{62}$$

Proof. Let $v = [\gamma(t)]$. Then $\mathrm{H}\mathcal{L}^h_*v = \mathrm{H}l^h_*v = [h\gamma(t)] - [e^{-t\boldsymbol{a}[h\gamma(t)]}e] = [h\gamma(t)] + [e^{t(l^{h^*}\boldsymbol{a})(v)}he] = [h\gamma(t)] + [he^{t\boldsymbol{a}(v)}e] + [e^{tT_A(h)(v)}he] = \mathcal{L}^h_*(v + [e^{t\boldsymbol{a}(v)}e]) = \mathcal{L}^h_*\mathrm{H}v$.

Curvature.

An n-form $\boldsymbol{\vartheta}$ is said to be horizontal if $\boldsymbol{\vartheta}(u_1, u_2, \ldots u_n) = 0$ whenever at least one of the vectors $u_i \in T_eE$ is vertical. The exterior covariant derivative $D\boldsymbol{\omega}$ of an n-form $\boldsymbol{\omega}$ is the $(n+1)$-horizontal form defined by

$$D\boldsymbol{\omega}(v_1, v_2, \ldots, v_{n+1}) \equiv d\boldsymbol{\omega}(\mathrm{H}v_1, \mathrm{H}v_2, \ldots, \mathrm{H}v_{n+1}) - (-1)^nT_A(\boldsymbol{\omega})(\mathrm{H}v_1, \mathrm{H}v_2, \ldots, \mathrm{H}v_{n+1}) \tag{63}$$

for all $v_i \in T_eE$ and $e \in E$. In the above formula $T_A(\boldsymbol{\omega})$ is defined by

$$T_A(\boldsymbol{\omega}) \equiv \boldsymbol{\omega}^\alpha \wedge T_{A*}(X^\alpha)\ , \tag{64}$$

where $T_{A*} : \mathrm{Lie}(H) \to \mathrm{Lie}(H) \otimes \Omega^1(E)$ is the differential of $T_A : H \to \mathrm{Lie}(H) \otimes \Omega^1(E)$. If H is normal in $Aut(H)$ we simply have $T_A(\boldsymbol{\omega}) = \boldsymbol{\omega}^\rho \wedge p^*A^\sigma[X^\rho, X^\sigma] = [\boldsymbol{\omega}, p^*A]$, where now X^ρ are generators of $\mathrm{Lie}(Aut(H))$.

The 2-curvature of the 2-connection $(\boldsymbol{a}, A)$ is given by the couple

$$(\boldsymbol{k}, K) \equiv \big(D\boldsymbol{a}\,,\, dA + A \wedge A\big). \tag{65}$$

We have the Cartan structural equation

$$\boldsymbol{k} = d\boldsymbol{a} + \frac{1}{2}[\boldsymbol{a}, \boldsymbol{a}] + T_A(\boldsymbol{a})\ , \tag{66}$$

where $\frac{1}{2}[\boldsymbol{a}, \boldsymbol{a}] = \frac{1}{2}\boldsymbol{a}^\alpha \wedge \boldsymbol{a}^\beta[X^\alpha, X^\beta] = \boldsymbol{a} \wedge \boldsymbol{a}$ with $X^\alpha \in \mathrm{Lie}(H)$,

The proof of eq. (66) is very similar to the usual proof of the Cartan structural equation for principal bundles. One has just to notice that the extra term $T_A(\boldsymbol{a})$ is necessary since $d\boldsymbol{a}(\mathrm{V}v, \mathrm{H}u) = -\boldsymbol{a}([\mathrm{V}v, \mathrm{H}u]) = T_{A*}(\boldsymbol{a}(\mathrm{V}v))(\mathrm{H}u) = -T_A(\boldsymbol{a})(\mathrm{V}v, \mathrm{H}u)$.

The 2-curvature $(\boldsymbol{k}, K)$ satisfies the following generalized equivariance property

$$l^{h^*}\boldsymbol{k} = Ad_h\boldsymbol{k} + T_K(h)\ , \tag{67}$$

where with abuse of notation we have written $T_K(h)$ instead of $T_{p^*K}(h)$. We also have the Bianchi identities, $dK + A \wedge K = 0$ and

$$D\boldsymbol{k} = 0\ . \tag{68}$$

Given an horizontal n-form $\boldsymbol{\vartheta}$ on E that is Θ-equivariant, i.e. that satisfies $l^{h^*}\boldsymbol{\vartheta} = Ad_h\boldsymbol{\vartheta} + T_\Theta(h)$, where Θ is an n-form on M, we have the structural equation

$$D\boldsymbol{\vartheta} = d\boldsymbol{\vartheta} + [\boldsymbol{a}, \boldsymbol{\vartheta}] + T_\Theta(\boldsymbol{a}) - (-1)^n T_A(\boldsymbol{\vartheta}) \tag{69}$$

where $[\boldsymbol{a}, \boldsymbol{\vartheta}] = \boldsymbol{a}^\alpha \wedge \boldsymbol{\vartheta}^\beta[X^\alpha, X^\beta] = \boldsymbol{a} \wedge \boldsymbol{\vartheta} - (-1)^n\boldsymbol{\vartheta} \wedge \boldsymbol{a}$. The proof is again similar to the usual one (where $\Theta = 0$) and is left to the reader. We also have that $D\boldsymbol{\vartheta}$ is $(d\Theta + [A, \Theta])$-equivariant,

$$l^{h^*}D\boldsymbol{\vartheta} = Ad_h\boldsymbol{\vartheta} + T_{d\Theta+[A,\Theta]}(h)\ . \tag{70}$$

Combining (69) and (68) we obtain the explicit expression of the Bianchi identity

$$d\boldsymbol{k} + [\boldsymbol{a}, \boldsymbol{k}] + T_K(\boldsymbol{a}) - T_A(\boldsymbol{k}) = 0 \tag{71}$$

We also have

$$D^2\boldsymbol{\vartheta} = [\boldsymbol{k}, \boldsymbol{\vartheta}] + T_\Theta(\boldsymbol{k}) - (-1)^n T_K(\boldsymbol{\vartheta})\ . \tag{72}$$

As was the case for the 2-connection $(\boldsymbol{a}, A)$, also for the 2-curvature $(\boldsymbol{k}, K)$ we can have a formulation using the right H action instead of the left one. Indeed one can prove that if $(\boldsymbol{k}, K)$ is a 2-curvature then $(\boldsymbol{k}^r, K^r)$ where

$$\boldsymbol{k}^r = -\boldsymbol{\varphi}^{-1}(\boldsymbol{k})\ ,\ \ K^r = \boldsymbol{\varphi}^{-1}(K + ad_{\boldsymbol{k}})\boldsymbol{\varphi}$$

is the right 2-curvature associated with the right 2-connection $(\boldsymbol{a}^r, A^r)$. In other words we have that $\boldsymbol{k}^r$ is horizontal and that

$$\boldsymbol{k}^r = \boldsymbol{k}_{\boldsymbol{a}^r}\ ,\ \ K^r = K_{A^r}$$

(for the proof we used $T_{A^r}(\boldsymbol{\varphi}^{-1}(X)) = \boldsymbol{\varphi}^{-1}([X, \boldsymbol{a}] + T_A(X)) + d\boldsymbol{\varphi}^{-1}(X)$, $X \in \mathrm{Lie}(H)$). We also have

$$r^{h^*}\boldsymbol{k}^r = Ad_{h^{-1}}\boldsymbol{k}^r + T_{K^r}(h^{-1})\ . \tag{73}$$

More in general consider the couple $(\boldsymbol{\vartheta}, \Theta)$ where $\boldsymbol{\vartheta}$, is an horizontal n-form on E that is Θ-equivariant. Then we have the couple $(\boldsymbol{\vartheta}^r, \Theta^r)$ where $\boldsymbol{\vartheta}^r = -\boldsymbol{\varphi}^{-1}(\boldsymbol{\vartheta})$ is an horizontal n-form on E that is right Θ^r-equivariant,

$$r^{h^*}\boldsymbol{\vartheta}^r = Ad_{h^{-1}}\boldsymbol{\vartheta}^r + T_{\Theta^r}(h^{-1}) \ . \tag{74}$$

with $\Theta^r = \boldsymbol{\varphi}^{-1}(\Theta + ad_{\boldsymbol{\vartheta}})\boldsymbol{\varphi}$.

The pull-back of a 2-connection (or of a horizontal form) on a principal H-bibundle is a 2-connection (horizontal form) on the pulled back principal H-bibundle, moreover the exterior covariant derivative -and in particular the definition of 2-curvature- commutes with the pull-back operation.

Local coordinates description.
Let's consider the sections $\boldsymbol{t}^i : U^i \to E$ subordinate to the covering $\{U^i\}$ of M. Let $\iota : H \times U^i \to p^{-1}(U^i) \subset E$ be the local trivialization of E induced by $\boldsymbol{t}^i$ according to $\iota(x,h) = h\boldsymbol{t}^i(x)$, where $x \in M$. We define the one-forms on $U^i \subset M$

$$a^i = \boldsymbol{t}^{i*}\boldsymbol{a} \ , \tag{75}$$

then, the local expression of $\boldsymbol{a}$ is $ha^ih^{-1} + T_A(h) + hdh^{-1}$, more precisely,

$$\iota^*(\boldsymbol{a})_{(x,h)}(v_M, v_H) = ha^i(x)h^{-1}(v_M) + T_{A(x)}(h)(v_M) + hdh^{-1}(v_H) \ , \tag{76}$$

where v_M ,v_H are respectively tangent vectors of $U^i \subset M$ at x, and of H at h, and where $-hdh^{-1}$ denotes the Maurer-Cartan one-form on H evaluated at $h \in H$. Similarly the local expression for $\boldsymbol{k}$ is $hk^ih^{-1} + T_K(h)$, where $k^i = \boldsymbol{t}^{i*}\boldsymbol{k}$.

Using the sections $\{\boldsymbol{t}^i\}$ we also obtain an explicit expression for A^r,

$$A^r = \boldsymbol{t}^{i*}\mathcal{A}^r = \varphi_i^{-1}(A + ad_{a^i})\varphi_i + \varphi_i^{-1}d\varphi_i \ . \tag{77}$$

Of course in U^{ij} we have $\boldsymbol{t}^{i*}\mathcal{A}^r = \boldsymbol{t}^{j*}\mathcal{A}^r$, so that A^r is defined on all M. In U^{ij} we also have $a^i = h^{ij}a^jh^{ij^{-1}} + h^{ij}dh^{ij^{-1}} + T_A(h^{ij})$ and $k^i = h^{ij}k^jh^{ij^{-1}} + T_K(h^{ij})$.

Sum of 2-connections.
If the group H is abelian, on the product bundle E_1E_2 there is the natural connection $\boldsymbol{a}_1 + \boldsymbol{a}_2$ obtained from the connections $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ on E_1 and E_2. In this subsection we generalize to the nonabelian case the sum of connections. Consider the following diagram

$$\begin{tikzcd} E_1 \oplus E_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[drr, "\pi_\oplus"] & E_2 & \\ E_1 & & E_1E_2 \end{tikzcd} \tag{78}$$

and let $(\boldsymbol{a}_1, A_2)$ be a 2-connection on E_1 and $(\boldsymbol{a}_2, A_2)$ a 2-connection on E_2. Recalling the definition of the product E_1E_2, we see that the one-form on $E_1 \oplus E_2$

$$\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) \tag{79}$$

is the pull-back of a one-form on E_1E_2 iff, for all $v_1 \in T_{e_1}E$, $v_2 \in T_{e_2}E$ and $h \in H$,

$$\begin{aligned} &\big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big)_{(e_1,e_2)}(v_1, v_2) \\ &\quad = \big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big)_{(e_1h^{-1},he_2)}(r_*^h v_1, l_*^h v_2) \\ &\qquad + \big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big)_{(e_1h^{-1},he_2)}([e_1h^{-1}(t)], [h(t)e_2]) \end{aligned}$$

where $h(t)$ is an arbitrary curve in H with $h(0) = 1_H$. Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ satisfy the Cartan-Maurer condition (39) the last addend vanishes identically and therefore the expression is equivalent to

$$\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) = rl^{h^*}\big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big) \tag{80}$$

where

$$\begin{aligned} rl^h \,:\, E_1 \oplus E_2 \quad &\to \quad E_1 \oplus E_2 \ , \\ (e_1, e_2) \quad &\mapsto \quad (e_1 h^{-1}, he_2) \ . \end{aligned}$$

Now, using (7), and then (52) we have

$$\begin{aligned} rl^{h^*}\big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big) &= \pi_1^* r^{h^{-1}*}\boldsymbol{a}_1 + \boldsymbol{\varphi}_1 Ad_{h^{-1}}(\pi_2^* l^{h^*}\boldsymbol{a}_2) \\ &= \pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) \\ &\quad + \boldsymbol{\varphi}_1\big(\pi_1^* T_{A_1^r}(h^{-1}) + \pi_2^* Ad_{h^{-1}} T_{A_2}(h)\big) \end{aligned}$$

and the last addend vanishes iff

$$A_2 = {A_1}^r \ . \tag{81}$$

In conclusion, when (81) holds, there exists a one-form on E_1E_2, denoted by $\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2$, such that

$$\pi_\oplus^*(\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2) = \pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) \tag{82}$$

From this expression it is easy to see that $(\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2, A_1)$ is a 2-connection on E_1E_2. We then say that $(\boldsymbol{a}_1, A_1)$ and $(\boldsymbol{a}_2, A_2)$ (or simply that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$) are *summable* and we write

$$(\boldsymbol{a}_1, A_1) \boldsymbol{+} (\boldsymbol{a}_2, A_2) = (\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2, A_1) \ . \tag{83}$$

Notice that the sum operation $\boldsymbol{+}$ thus defined is associative (and noncommutative). In other words, if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are summable, and if $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are summable then $\boldsymbol{a}_1 \boldsymbol{+} (\boldsymbol{a}_2 \boldsymbol{+} \boldsymbol{a}_3) = (\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2) \boldsymbol{+} \boldsymbol{a}_3$ and $(\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2 \boldsymbol{+} \boldsymbol{a}_3, A_1)$ is a 2-connection on $E_1E_2E_3$.

We also have a summability criterion for the couples $(\boldsymbol{\vartheta}_1, \Theta_1)$ and $(\boldsymbol{\vartheta}_2, \Theta_2)$ where $\boldsymbol{\vartheta}_i$, $i = 1, 2$ is an horizontal n-form on E_i that is Θ_i-equivariant. We have that $(\boldsymbol{\vartheta}_1, \Theta_1) \boldsymbol{+} (\boldsymbol{\vartheta}_2, \Theta_2) = (\boldsymbol{\vartheta}_1 \boldsymbol{+} \boldsymbol{\vartheta}_2, \Theta_1)$ where

$$\pi_\oplus^*(\boldsymbol{\vartheta}_1 \boldsymbol{+} \boldsymbol{\vartheta}_2) = \pi_1^*\boldsymbol{\vartheta}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{\vartheta}_2) \tag{84}$$

is a well defined horizontal Θ_1-equivariant n-form on E_1E_2 iff

$$\Theta_2 = {\Theta_1}^r \ . \tag{85}$$

We have

$$(D_{\boldsymbol{a}_1}\boldsymbol{\vartheta}_1, D_{A_1}\Theta_1) \boldsymbol{+} (D_{\boldsymbol{a}_2}\boldsymbol{\vartheta}_2, D_{A_2})\Theta_2 = (D_{\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2}(\boldsymbol{\vartheta}_1 \boldsymbol{+} \boldsymbol{\vartheta}_2), D_{A_1}\Theta_1), \tag{86}$$

with obvious notation: $D_{\boldsymbol{a}}\boldsymbol{\vartheta} = d\boldsymbol{\vartheta} + [\boldsymbol{a}, \boldsymbol{\vartheta}] + T_\Theta(\boldsymbol{a}) - (-1)^n T_A(\boldsymbol{\vartheta})$ and $D_A\Theta = d\Theta + [A, \Theta]$. Also the summability of curvatures is a direct consequence of the summability of their corresponding connections. If $(\boldsymbol{a}_1, A_1) \boldsymbol{+} (\boldsymbol{a}_2, A_2) = (\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2, A_1)$ then

$$(\boldsymbol{k}_1, K_1) \boldsymbol{+} (\boldsymbol{k}_2, K_2) = (\boldsymbol{k}_1 \boldsymbol{+} \boldsymbol{k}_2, K_1) \ , \tag{87}$$

and we also have

$$\boldsymbol{k}_{a_1 \boldsymbol{+} a_2} = \boldsymbol{k}_1 \boldsymbol{+} \boldsymbol{k}_2 \ . \tag{88}$$

Summability is preserved under isomorphism, i.e. if $\boldsymbol{a}_i$ are summable connections on E_i $(i = 1, 2)$ and we have isomorphisms $\sigma_i \,:\, E_i' \to E_i$, then $\sigma_i^*(\boldsymbol{a}_i)$ are summable and $\sigma_1^*(\boldsymbol{a}_2) \boldsymbol{+} \sigma_2^*(\boldsymbol{a}_2) = \sigma_{12}^*(\boldsymbol{a}_1 \boldsymbol{+} \boldsymbol{a}_2)$, where we have considered the induced isomorphism $\sigma_{12} \equiv \sigma_1\sigma_2 \,:\, E_1'E_2' \to E_1E_2$. The same property holds for horizontal forms.

4. **Nonabelian Bundle Gerbes**

Now that we have the notion of product of principal bibundles we can define nonabelian bundle gerbes generalizing the construction studied by Murray [5] (see also Hitchin [3] and [4]) in the abelian case.

Consider a submersion $\wp\ :\ Y \to M$ (i.e. a map onto with differential onto) we can always find a covering $\{O_\alpha\}$ of M with local sections $\sigma_\alpha\ :\ O_\alpha \to Y$, i.e. $\wp \circ \sigma_\alpha = id$. The manifold Y will always be equipped with the submersion $\wp\ :\ Y \to M$. We also consider $Y^{[n]} = Y \times_M Y \times_M Y \ldots \times_M Y$ the n-fold fiber product of Y, i.e. $Y^{[n]} \equiv \{(y_1, \ldots y_n) \in Y^n \mid \wp(y_1) = \wp(y_2) = \ldots \wp(y_n)\}$.

Given a H principal bibundle $\mathcal{E}$ over $Y^{[2]}$ we denote by $\mathcal{E}_{12} = p_{12}^*(\mathcal{E})$ the H principal bibundle on $Y^{[3]}$ obtained as pull-back of $p_{12}\ :\ Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{E}_{13}$ and $\mathcal{E}_{23}$.

Consider the quadruple $(\mathcal{E}, Y, M, \boldsymbol{f})$ where the H principal bibundle on $Y^{[3]}$, $\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}$ is trivial, and $\boldsymbol{f}$ is a global central section of $\left(\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}\right)^{-1}$ [i.e. $\boldsymbol{f}$ satisfies (3)]. Recalling the paragraph after formula (15) we can equivalently say that $\mathcal{E}_{12}\mathcal{E}_{23}$ and $\mathcal{E}_{13}$ are isomorphic, the isomorphism being given by the global central section $\boldsymbol{f}^{-1}$ of

$$\mathcal{T} \equiv \mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}\ . \tag{89}$$

We now consider $Y^{[4]}$ and the bundles $\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{13}, \mathcal{E}_{24}, \mathcal{E}_{34}, \mathcal{E}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12}\ :\ Y^{[4]} \to Y^{[2]}$ etc., and $\mathcal{T}_{123}^{-1}, \mathcal{T}_{124}^{-1}, \mathcal{T}_{134}^{-1}$ relative to $p_{123}\ :\ Y^{[4]} \to Y^{[3]}$ etc.. Since the product of bundles commutes with the pull-back of bundles, we then have

$$\mathcal{T}_{124}^{-1}\mathcal{E}_{12}(\mathcal{T}_{234}^{-1}\mathcal{E}_{23}\mathcal{E}_{34}) = \mathcal{T}_{134}^{-1}(\mathcal{T}_{123}^{-1}\mathcal{E}_{12}\mathcal{E}_{23})\mathcal{E}_{34} = \mathcal{E}_{14} \tag{90}$$

as bundles on $Y^{[4]}$. The first identity in (90) is equivalent to

$$\mathcal{T}_{124}^{-1}\,\mathcal{E}_{12}\mathcal{T}_{234}^{-1}\mathcal{E}_{12}^{-1} = \mathcal{T}_{134}^{-1}\,\mathcal{T}_{123}^{-1} \tag{91}$$

Let us now consider the global central section $\boldsymbol{f}$ of $\mathcal{T}^{-1} = \mathcal{E}_{13}\mathcal{E}_{23}^{-1}\mathcal{E}_{12}^{-1}$ and denote by $\boldsymbol{f}_{124}$ ($\boldsymbol{f}_{234}$, etc.) the global central section of $\mathcal{T}_{124}^{-1}$ ($\mathcal{T}_{234}^{-1}$, etc.) obtained as the pull-back of $\boldsymbol{f}$. Consistently with (91) we can require the condition

$$\boldsymbol{f}_{124}\,\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234}) = \boldsymbol{f}_{134}\,\boldsymbol{f}_{123} \tag{92}$$

where, following the notation of (27), $\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234})$ is the section of $\mathcal{T}_{234}^{-1}$ that in any open $\mathcal{U} \subset Y^{[4]}$ equals $\boldsymbol{s}_{12}\boldsymbol{f}_{234}\boldsymbol{s}_{12}^{-1}$ where $\boldsymbol{s}_{12}\ :\ \mathcal{U} \to \mathcal{E}_{12}$ is any section of $\mathcal{E}_{12}$, in particular we can choose $\boldsymbol{s}_{12}$ to be the pull-back of a section $\boldsymbol{s}$ of $\mathcal{E}$.

Definition 11. *A Bundle gerbe $\mathcal{G}$ is the quadruple $(\mathcal{E}, Y, M, \boldsymbol{f})$ where the H principal bibundle on $Y^{[3]}$, $\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}$ is trivial and $\boldsymbol{f}$ is a global central section of $\left(\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}\right)^{-1}$ that satisfies (92).*

Recall that when H has trivial centre then the section $\boldsymbol{f}$ of $\mathcal{T}^{-1}$ is unique; it then follows that relation (92) is automatically satisfied because the bundle on the l.h.s. and the bundle on the r.h.s. of (91) admit just one global central section, respectively $\boldsymbol{f}_{124}\,\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234})$ and $\boldsymbol{f}_{134}\,\boldsymbol{f}_{123}$. Therefore, if H has trivial centre, a bundle gerbe $\mathcal{G}$ is simply the triple $(\mathcal{E}, Y, M)$, where $\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}$ is trivial.

Consider an H principal bibundle N over Z an let $\mathcal{N}_1 = p_1^*(N)$, $\mathcal{N}_2 = p_2^*(N)$, be the pull-back of N obtained respectively from $p_1\ :\ Z^{[2]} \to Z$ and $p_2\ :\ Z^{[2]} \to Z$ (p_1

projects on the first component, p_2 on the second). If $(\mathcal{E}, Z, M, \boldsymbol{f})$ is a bundle gerbe also $\big(\mathcal{N}_1\mathcal{E}\mathcal{N}_2^{-1}, Z, M, \boldsymbol{\varphi}_1(\boldsymbol{f})\big)$ is a bundle gerbe. Here $\boldsymbol{\varphi}_1(\boldsymbol{f})$ is the canonical global central section of the bibundle $\mathcal{N}_1\mathcal{T}^{-1}\mathcal{N}_1^{-1}$ and now $\mathcal{N}_1$ is the pull-back of N via $p_1 : Z^{[3]} \to Z$; locally $\boldsymbol{\varphi}_1(\boldsymbol{f}) = \boldsymbol{s}_1\boldsymbol{f}\boldsymbol{s}_1^{-1}$ where $\boldsymbol{s}_1$ is the pull-back of any local section $\boldsymbol{s}$ of N. Similarly also $\big(\eta\mathcal{E}, Z, M, \boldsymbol{\ell}_{13}^{-1}\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}\big)$ is a bundle gerbe if η^{-1} is a trivial bundle on $Z^{[2]}$ with global central section $\boldsymbol{\ell}$ (as usual $\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})$ denotes the canonical section of $\mathcal{E}_{12}\eta_{23}^{-1}\mathcal{E}_{12}^{-1}$). This observations lead to the following definition [36]

Definition 12. *Two bundle gerbes $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ and $\mathcal{G}' = (\mathcal{E}', Y', M, \boldsymbol{f}')$ are stably isomorphic if there exists a bibundle $\mathcal{N}$ over $Z = Y \times_{M} Y'$ and a trivial bibundle η^{-1} over $Z^{[2]}$ with section $\boldsymbol{\ell}$ such that*

$$\mathcal{N}_1\, q'^*\mathcal{E}'\, \mathcal{N}_2^{-1} = \eta\, q^*\mathcal{E} \tag{93}$$

and

$$\boldsymbol{\varphi}_1(q'^*\boldsymbol{f}') = \boldsymbol{\ell}_{13}^{-1} q^*\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12} \tag{94}$$

where $q^*\mathcal{E}$ and $q'^*\mathcal{E}'$ are the pull-back bundles relative to the projections $q : Z^{[2]} \to Y^{[2]}$ and $q' : Z^{[2]} \to Y'^{[2]}$. Similarly $q'^*\boldsymbol{f}'$ and $q^*\boldsymbol{f}$ are the pull-back sections relative to the projections $q : Z^{[3]} \to Y^{[3]}$ and $q' : Z^{[3]} \to Y'^{[3]}$.

The relation of stable isomorphism is an equivalence relation.

The bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ is called trivial if it is stably isomorphic to the trivial bundle gerbe $(Y \times H, Y, M, \mathbf{1})$; we thus have that $\mathcal{E}$ and $\mathcal{N}_1^{-1}\mathcal{N}_2$ are isomorphic as H-bibundles, i.e.

$$\mathcal{E} \sim \mathcal{N}_1^{-1}\mathcal{N}_2 \tag{95}$$

and that $\boldsymbol{f} = \boldsymbol{\varphi}_1^{-1}(\boldsymbol{\ell}_{13}^{-1}\boldsymbol{\ell}_{23}\boldsymbol{\ell}_{12})$ where $\boldsymbol{\ell}$ is the global central section of $\eta^{-1} \equiv \mathcal{N}_2\mathcal{E}^{-1}\mathcal{N}_1^{-1}$.

Proposition 13. *Consider a bundle gerbe $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ with submersion $\wp : Y \to M$; a new submersion $\wp' : Y' \to M$ and a (smooth) map $\sigma : Y' \to Y$ compatible with $\wp$ and $\wp'$ (i.e. $\wp \circ \sigma = \wp'$). The pull-back bundle gerbe $\sigma^*\mathcal{G}$ (with obvious abuse of notation) is given by $(\sigma^*\mathcal{E}, Y', M, \sigma^*\boldsymbol{f})$. We have that the bundle gerbes $\mathcal{G}$ and $\sigma^*\mathcal{G}$ are stably equivalent.*

Proof. Consider the following identity on $Y^{[4]}$:

$$\mathcal{E}_{11'}\mathcal{E}_{1'2'}\mathcal{E}_{22'}^{-1} = \eta_{12}\mathcal{E}_{12} \tag{96}$$

where $\eta_{12} = \mathcal{T}_{11'2'}\mathcal{T}_{122'}^{-1}$ so that η_{12}^{-1} has section $\boldsymbol{\ell}_{12} = \boldsymbol{f}_{122'}^{-1}\boldsymbol{f}_{11'2'}$; the labelling $1, 1', 2, 2'$ instead of $1, 2, 3, 4$ is just a convention. Multiplying three times (96) we obtain the following identity between trivial bundles on $Y^{[6]}$ $\mathcal{E}_{11'}\mathcal{T}_{1'2'3'}\mathcal{E}_{11'}^{-1} = \eta_{12}\,\mathcal{E}_{12}\eta_{23}\mathcal{E}_{12}^{-1}\,\mathcal{T}_{123}\,\eta_{13}^{-1}$. The sections of (the inverses of) these bundles satisfy

$$\boldsymbol{\varphi}_{11'}(\boldsymbol{f}') = \boldsymbol{\ell}_{13}^{-1}\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}\ , \tag{97}$$

thus $\mathcal{E}_{1'2'}$ and $\mathcal{E}_{12}$ give stably equivalent bundle gerbes. Next we pull-back the bundles in (96) using $(id, \sigma, id, \sigma) : Z^{[2]} \to Y^{[4]}$ where $Z = Y \times_M Y'$; recalling that the product commutes with the pull-back we obtain relation (93) with $\eta = (id, \sigma, id, \sigma)^*\eta_{12}$ and $N = (id, \sigma)^*\mathcal{E}$. We also pull-back (97) with $(id, \sigma, id, \sigma, id, \sigma) : Z^{[3]} \to Y^{[6]}$ and obtain formula (94). $\square$

Theorem 14. *Locally a bundle gerbe is always trivial: $\forall x \in M$ there is an open O of x such that the bundle gerbe restricted to O is stably isomorphic to the trivial bundle gerbe*

$(Y|_O^{[2]} \times H, Y|_O, O, \mathbf{1})$. Here $Y|_O$ is Y restricted to O: $Y|_O = \{y \in Y \mid \wp(y) \in O \subset M\}$. Moreover in any sufficiently small open $\mathcal{U}$ of $Y|_O^{[3]}$ one has

$$\boldsymbol{f} = \boldsymbol{s''}_{13}\boldsymbol{s'}^{-1}_{23}\boldsymbol{s}^{-1}_{12} \tag{98}$$

with $\boldsymbol{s}^{-1}_{12}, \boldsymbol{s'}^{-1}_{23}$ and $\boldsymbol{s''}_{13}$ respectively sections of $\mathcal{E}^{-1}_{12}, \mathcal{E}^{-1}_{23}$ and $\mathcal{E}_{13}$ that are pull-backs of sections of $\mathcal{E}$.

Proof. Choose $O \subset M$ such that there exists a section $\sigma : O \to Y|_O$. Define the maps

$$\begin{aligned} r_{[n]} &: Y|_O^{[n]} \to Y|_O^{[n+1]} \\ (y_1, \ldots y_n) &\mapsto (y_1, \ldots y_n, \sigma(\wp(y_n))) \end{aligned}$$

notice that $\sigma(\wp(y_1)) = \sigma(\wp(y_2)) \ldots = \sigma(\wp(y_n))$. It is easy to check the following equalities between maps on $Y|_O^{[2]}$, $p_{12} \circ r_{[2]} = id$, $p_{13} \circ r_{[2]} = r_{[1]} \circ p_1$, $p_{23} \circ r_{[2]} = r_{[1]} \circ p_2$, and between maps on $Y|_O^{[3]}$

$$p_{123} \circ r_{[3]} = id \ , \ p_{124} \circ r_{[3]} = r_{[2]} \circ p_{12} \ , \ p_{234} \circ r_{[3]} = r_{[2]} \circ p_{23} \ , \ p_{134} \circ r_{[3]} = r_{[2]} \circ p_{13} \ . \tag{99}$$

We now pull back with $r_{[2]}$ the identity $\mathcal{E}_{12} = \mathcal{T}\mathcal{E}_{13}\mathcal{E}^{-1}_{23}$ and obtain the following local trivialization of $\mathcal{E}$

$$\mathcal{E} = r^*_{[2]}(\mathcal{T})\,\mathcal{N}_1\mathcal{N}_2^{-1}$$

where $\mathcal{N}_1 = p_1^*(N)$, $\mathcal{N}_2 = p_2^*(N)$ and $N = r^*_{[1]}(\mathcal{E})$. Let $\mathcal{U} = U \times_O U' \times_O U'' \subset Y|_O^{[3]}$ where U, U', U'' are opens of $Y|_O$ that respectively admit the sections $\boldsymbol{n} : U \to N$, $\boldsymbol{n'} : U' \to N$, $\boldsymbol{n''} : U'' \to N$. Consider the local sections $\boldsymbol{s} = r^*_{[2]}(\boldsymbol{f}^{-1})\boldsymbol{n}_1\boldsymbol{n'}^{-1}_2 : U \times_O U' \to \mathcal{E}$, $\boldsymbol{s'} = r^*_{[2]}(\boldsymbol{f}^{-1})\boldsymbol{n'}_2\boldsymbol{n''}^{-1}_3 : U' \times_O U'' \to \mathcal{E}$, $\boldsymbol{s''} = r^*_{[2]}(\boldsymbol{f}^{-1})\boldsymbol{n}_1\boldsymbol{n''}^{-1}_3 : U \times_O U'' \to \mathcal{E}$ and pull them back to local sections $\boldsymbol{s}_{12}$ of $\mathcal{E}_{12}$, $\boldsymbol{s'}_{23}$ of $\mathcal{E}_{23}$ and $\boldsymbol{s''}_{13}$ of $\mathcal{E}_{13}$. Then (98) holds because, using (99), the product $\boldsymbol{s''}_{13}\boldsymbol{s'}^{-1}_{23}\boldsymbol{s}^{-1}_{12}$ equals the pull-back with $r_{[3]}$ of the section $\boldsymbol{f}^{-1}_{134}\boldsymbol{f}_{124}\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234}) = \boldsymbol{f}_{123}$ [cf. (92)]. $\square$

Local description.

Locally we have the following description of a bundle gerbe; we choose an atlas of charts for the bundle $\mathcal{E}$ on $Y^{[2]}$, i.e. sections $\boldsymbol{t}^i : \mathcal{U}^i \to \mathcal{E}$ relative to a trivializing covering $\{\mathcal{U}^i\}$ of $Y^{[2]}$. We write $\mathcal{E} = \{h^{ij}, \varphi^i\}$. We choose also atlases for the pull-back bundles $\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{13}$; we write $\mathcal{E}_{12} = \{h^{ij}_{12}, \varphi^i_{12}\}$, $\mathcal{E}_{23} = \{h^{ij}_{23}, \varphi^i_{23}\}$, $\mathcal{E}_{13} = \{h^{ij}_{13}, \varphi^i_{13}\}$, where these atlases are relative to a common trivializing covering $\{\mathcal{U}^i\}$ of $Y^{[3]}$. It then follows that $\mathcal{T} = \{f^i f^{j^{-1}}, Ad_{f^i}\}$ where $\{f^{i^{-1}}\}$ are the local representatives for the section $\boldsymbol{f}^{-1}$ of $\mathcal{T}$. We also consider atlases for the bundles on $Y^{[4]}$ that are relative to a common trivializing covering $\{\mathcal{U}^i\}$ of $Y^{[4]}$ (with abuse of notation we denote with the same index i all these different coverings[1]). Then (89), that we rewrite as $\mathcal{E}_{12}\mathcal{E}_{23} = \mathcal{T}\mathcal{E}_{13}$, reads

$$h^{ij}_{12}\varphi^j_{12}(h^{ij}_{23}) = f^i h^{ij}_{13} f^{j^{-1}} \ , \ \varphi^i_{12} \circ \varphi^i_{23} = Ad_{f^i} \circ \varphi^i_{13} \tag{100}$$

[1] An explicit construction is for example obtained pulling back the atlas of $\mathcal{E}$ to the pull-back bundles on $Y^{[3]}$ and on $Y^{[4]}$. The sections $\boldsymbol{t}^i : \mathcal{U}^i \to \mathcal{E}$ induce the associated sections $\boldsymbol{t}^i_{12} \equiv p^*_{12}(\boldsymbol{t}^i) : \mathcal{U}^i_{12} \to \mathcal{E}_{12}$ where $p_{12} : Y^{[3]} \to Y^{[2]}$ and $\mathcal{U}^i_{12} \equiv p^{-1}_{12}(\mathcal{U}^i)$. We then have $\mathcal{E}_{12} = \{h^{ij}_{12}, \varphi^i_{12}\}$ with $h^{ij}_{12} = p^*_{12}(h^{ij})$, $\varphi^i_{12} = p^*_{12}(\varphi^i)$. Similarly for $\mathcal{E}_{13}$, $\mathcal{E}_{23}$. The $Y^{[3]}$ covering given by the opens $\mathcal{U}^I \equiv \mathcal{U}^{ii'i''} \equiv \mathcal{U}^i_{12} \cap \mathcal{U}^{i'}_{23} \cap \mathcal{U}^{i''}_{13}$ can then be used for a common trivialization of the $\mathcal{E}_{12}$, $\mathcal{E}_{13}$ and $\mathcal{E}_{23}$ bundles; the respective sections are $\boldsymbol{t}^I_{12} = \boldsymbol{t}^i_{12}|_{\mathcal{U}^I}$, $\boldsymbol{t}^I_{23} = \boldsymbol{t}^{i'}_{23}|_{\mathcal{U}^I}$, $\boldsymbol{t}^I_{13} = \boldsymbol{t}^{i''}_{13}|_{\mathcal{U}^I}$; similarly for the transition functions $h^I_{12}, h^I_{23}, h^I_{13}$ and for $\varphi^I_{12}, \varphi^I_{23}, \varphi^I_{13}$. In $\mathcal{U}^I$ we then have $\boldsymbol{f}^{-1} = f^{I^{-1}}\boldsymbol{t}^I_{12}\boldsymbol{t}^I_{23}\boldsymbol{t}^{I^{-1}}_{13}$.

and relation (92) reads

$$\varphi^i_{12}(f^i_{234})f^i_{124} = f^i_{123}f^i_{134}\ . \tag{101}$$

Bundles and local data on M.
Up to equivalence under stable isomorphisms, there is an alternative geometric description of bundle gerbes, in terms of bundles on M. Consider the sections $\sigma_\alpha\ :\ O_\alpha \to Y$, relative to a covering $\{O_\alpha\}$ of M and consider also the induced sections $(\sigma_\alpha, \sigma_\beta)\ :\ O_{\alpha\beta} \to Y^{[2]}$, $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)\ :\ O_{\alpha\beta\gamma} \to Y^{[3]}$. Denote by $\mathcal{E}_{\alpha\beta}$, $\mathcal{T}_{\alpha\beta\gamma}$ the pull-back of the H-bibundles $\mathcal{E}$ and $\mathcal{T}$ via $(\sigma_\alpha, \sigma_\beta)$ and $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)$. Denote also by $\boldsymbol{f}_{\alpha\beta\gamma}$ the pull-back of the section $\boldsymbol{f}$. Then, following Hitchin description of abelian gerbes,

Definition 15. *A gerbe is a collection $\{\mathcal{E}_{\alpha\beta}\}$ of H principal bibundles $\mathcal{E}_{\alpha\beta}$ on each $O_{\alpha\beta}$ such that on the triple intersections $O_{\alpha\beta\gamma}$ the product bundles $\mathcal{E}_{\alpha\beta}\mathcal{E}_{\beta\gamma}\mathcal{E}^{-1}_{\alpha\gamma}$ are trivial, and such that on the quadruple intersections $O_{\alpha\beta\gamma\delta}$ we have $\boldsymbol{f}_{\alpha\beta\delta}\boldsymbol{\varphi}_{\alpha\beta}(\boldsymbol{f}_{\beta\gamma\delta}) = \boldsymbol{f}_{\alpha\gamma\delta}\boldsymbol{f}_{\alpha\beta\gamma}$.*

We also define two gerbes, given respectively by $\{\mathcal{E}'_{\alpha\beta}\}$ and $\{\mathcal{E}_{\alpha\beta}\}$ (we can always consider a common covering $\{O_\alpha\}$ of M), to be stably equivalent if there exist bibundles $\mathcal{N}_\alpha$ and trivial bibundles $\eta_{\alpha\beta}$ with (global central) sections $\boldsymbol{\ell}^{-1}_{\alpha\beta}$ such that

$$\mathcal{N}_\alpha\mathcal{E}'_{\alpha\beta}\mathcal{N}^{-1}_\beta = \eta_{\alpha\beta}\mathcal{E}_{\alpha\beta}\ , \tag{102}$$

$$\boldsymbol{\varphi}_\alpha(\boldsymbol{f}'_{\alpha\beta\gamma}) = \boldsymbol{\ell}^{-1}_{\alpha\gamma}\boldsymbol{f}_{\alpha\beta\gamma}\boldsymbol{\varphi}_{\alpha\beta}(\boldsymbol{\ell}_{\beta\gamma})\boldsymbol{\ell}_{\alpha\beta}\ . \tag{103}$$

A local description of the $\mathcal{E}_{\alpha\beta}$ bundles in terms of the local data (100), (101) can be given considering the refinement $\{O^i_\alpha\}$ of the $\{O_\alpha\}$ cover of M such that $(\sigma_\alpha, \sigma_\beta)(O^{ij}_{\alpha\beta}) \subset \mathcal{U}^{ij} \subset Y^{[2]}$, the refinement $\{O^i_\alpha\}$ such that $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)(O^{ijk}_{\alpha\beta\gamma}) \subset \mathcal{U}^{ijk} \subset Y^{[3]}$, and similarly for $Y^{[4]}$. We can then define the local data on M

$$h^{ij}_{\alpha\beta}\ :\ O^{ij}_{\alpha\beta} \to H \qquad \varphi^i_{\alpha\beta}\ :\ O^i_{\alpha\beta} \to Aut(H)$$

$$h^{ij}_{\alpha\beta} = h^{ij}_{12} \circ (\sigma_\alpha, \sigma_\beta) \quad \varphi^i_{\alpha\beta} = \varphi^i_{12} \circ (\sigma_\alpha, \sigma_\beta) \tag{104}$$

and

$$f^i_{\alpha\beta\gamma}\ :\ O^i_{\alpha\beta\gamma} \to H$$

$$f^i_{\alpha\beta\gamma} = f^i \circ (\sigma_\alpha, \sigma_\beta, \sigma_\gamma)\ . \tag{105}$$

It follows that $\mathcal{E}_{\alpha\beta} = \{h^{ij}_{\alpha\beta}, \varphi^i_{\alpha\beta}\}$ and $\mathcal{T}_{\alpha\beta\gamma} = \{f^i_{\alpha\beta\gamma}f^{j\ -1}_{\alpha\beta\gamma}, Ad_{f^i_{\alpha\beta\gamma}}\}$. Moreover relations (100), (101) imply the relations between local data on M

$$h^{ij}_{\alpha\beta}\varphi^j_{\alpha\beta}(h^{ij}_{\beta\gamma}) = f^i_{\alpha\beta\gamma}h^{ij}_{\alpha\gamma}f^{j\ -1}_{\alpha\beta\gamma}\ , \tag{106}$$

$$\varphi^i_{\alpha\beta} \circ \varphi^i_{\beta\gamma} = Ad_{f^i_{\alpha\beta\gamma}} \circ \varphi^i_{\alpha\gamma}\ , \qquad \varphi^i_{\alpha\beta}(f^i_{\beta\gamma\delta})f^i_{\alpha\beta\delta} = f^i_{\alpha\beta\gamma}f^i_{\alpha\gamma\delta}\ . \tag{107}$$

We say that (107) define a nonabelian Čech 2-cocycle. From (102), (103) we see that two sets $\{h^{ij}_{\alpha\beta}, \varphi^i_{\alpha\beta}, f^i_{\alpha\beta\gamma}\}$, $\{h'^{ij}_{\alpha\beta}, \varphi'^i_{\alpha\beta}, f'^i_{\alpha\beta\gamma}\}$ of local data on M are stably isomorphic if

$$h^{ij}_\alpha\varphi^j_\alpha(h'^{ij}_{\alpha\beta})\,\varphi^j_\alpha\varphi'^j_{\alpha\beta}\varphi^{j\ -1}_\beta(h^{ij}_\beta) = \ell^i_{\alpha\beta}\,h^{ij}_{\alpha\beta}\,\ell^{j\ -1}_{\alpha\beta}\ , \tag{108}$$

$$\varphi^i_\alpha \circ \varphi'^i_{\alpha\beta} \circ \varphi^{i\ -1}_\beta = Ad_{\ell^i_{\alpha\beta}} \circ \varphi^i_{\alpha\beta}\ , \tag{109}$$

$$\varphi^i_\alpha(f'^i_{\alpha\beta\gamma}) = \ell^i_{\alpha\beta}\,\varphi^i_{\alpha\beta}(\ell^i_{\beta\gamma})f^i_{\alpha\beta\gamma}\ell^{i\ -1}_{\alpha\gamma}\ , \tag{110}$$

here $\mathcal{N}_\alpha = \{h^{ij}_\alpha, \varphi_\alpha\}$, $\mathcal{E}'_{\alpha\beta} = \{h'^{ij}_{\alpha\beta}, \varphi'_{\alpha\beta}\}$ and $\eta_{\alpha\beta} = \{\ell^i_{\alpha\beta}\ell^{j\ -1}_{\alpha\beta}, Ad_{\ell^i_{\alpha\beta}}\}$.

We now compare the gerbe $\{\mathcal{E}_{\alpha\beta}\}$ obtained from a bundle gerbe $\mathcal{G}$ using the sections $\sigma_\alpha : O_\alpha \to Y$ to the gerbe $\{\mathcal{E}'_{\alpha\beta}\}$ obtained from $\mathcal{G}$ using a different choice of sections $\sigma'_\alpha : O_\alpha \to Y$. We first pull back the bundles in (96) using $(\sigma_\alpha, \sigma'_\alpha, \sigma_\beta, \sigma'_\beta) : O_{\alpha\beta} \to Y^{[4]}$; recalling that the product commutes with the pull-back we obtain the following relation between bundles respectively on O_α, $O_{\alpha\beta}$, O_β and on $O_{\alpha\beta}$, $O_{\alpha\beta}$,

$$\mathcal{N}_\alpha \mathcal{E}'_{\alpha\beta} \mathcal{N}_\beta^{-1} = \eta_{\alpha\beta} \mathcal{E}_{\alpha\beta} \ ,$$

here $\mathcal{N}_\alpha$ equals the pull-back of $\mathcal{E}_{11'}$ with $(\sigma_\alpha, \sigma'_\alpha) : O_\alpha \to Y^{[2]}$. We then pull back (97) with $(\sigma_\alpha, \sigma'_\alpha, \sigma_\beta, \sigma'_\beta, \sigma_\gamma, \sigma'_\gamma) : O_{\alpha\beta\gamma} \to Y^{[6]}$ and obtain formula (103). Thus $\{\mathcal{E}'_{\alpha\beta}\}$ and $\{\mathcal{E}_{\alpha\beta}\}$ are stably equivalent gerbes. We have therefore shown that the equivalence class of a gerbe (defined as a collection of bundles on $O_{\alpha\beta} \subset M$) is independent from the choice of sections $\sigma_\alpha : O_\alpha \to Y$ used to obtain it as pull-back from a bundle gerbe.

It is now easy to prove that equivalence classes of bundle gerbes are in one to one correspondence with equivalence classes of gerbes $\{\mathcal{E}_{\alpha\beta}\}$, and therefore with equivalence classes of local data on M. First of all we observe that a bundle gerbe $\mathcal{G}$ and its pull-back $\sigma^*\mathcal{G} = (\sigma^*\mathcal{E}, Y', M, \sigma^*\boldsymbol{f})$ (cf. Theorem 13) give the same gerbe $\{\mathcal{E}_{\alpha\beta}\}$ if we use the sections $\sigma'_\alpha : O_\alpha \to Y'$ for $\sigma^*\mathcal{G}$ and the sections $\sigma \circ \sigma'_\alpha : O_\alpha \to Y$ for $\mathcal{G}$. It then follows that two stably equivalent bundle gerbes give two stably equivalent gerbes. In order to prove the converse we associate to each gerbe $\{\mathcal{E}_{\alpha\beta}\}$ a bundle gerbe and then we prove that on equivalence classes this operation is the inverse of the operation $\mathcal{G} \to \{\mathcal{E}_{\alpha\beta}\}$. Given $\{\mathcal{E}_{\alpha\beta}\}$ we consider $Y = \sqcup O_\alpha$, the disjoint union of the opens $O_\alpha \subset M$, with projection $\wp(x, \alpha) = x$. Then $Y^{[2]}$ is the disjoint union of the opens $O_{\alpha\beta}$, i.e. $Y^{[2]} = \sqcup O_{\alpha\beta} = \cup O_{\alpha,\beta}$, where $O_{\alpha,\beta} = \{(\alpha, \beta, x) \,/\, x \in O_{\alpha\beta}\}$, similarly $Y^{[3]} = \sqcup O_{\alpha\beta\gamma} = \cup O_{\alpha,\beta,\gamma}$ etc.. We define $\mathcal{E}$ such that $\mathcal{E}|_{O_{\alpha,\beta}} = \mathcal{E}_{\alpha\beta}$ and we define the section $\boldsymbol{f}^{-1}$ of $\mathcal{T} = \mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1}$ to be given by $\boldsymbol{f}^{-1}|_{O_{\alpha,\beta,\gamma}} = \boldsymbol{f}^{-1}_{\alpha\beta\gamma}$, thus (92) holds. We write $(\sqcup\mathcal{E}_{\alpha\beta}, \sqcup O_\alpha, M, \sqcup\boldsymbol{f}_{\alpha\beta\gamma})$ for this bundle gerbe. If we pull it back with $\sigma_\alpha : O_\alpha \to Y$, $\sigma_\alpha(x) = (x, \alpha)$ we obtain the initial gerbe $\{\mathcal{E}_{\alpha\beta}\}$. In order to conclude the proof we have to show that $(\sqcup\mathcal{E}_{\alpha\beta}, \sqcup O_\alpha, M, \sqcup\boldsymbol{f}_{\alpha\beta\gamma})$ is stably isomorphic to the bundle gerbe $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ if $\{\mathcal{E}_{\alpha\beta}\}$ is obtained from $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ and sections $\sigma_\alpha : O_\alpha \to Y$. This holds because $(\sqcup\mathcal{E}_{\alpha\beta}, \sqcup O_\alpha, M, \sqcup\boldsymbol{f}_{\alpha\beta\gamma}) = \sigma^*\mathcal{G}$ with $\sigma : \sqcup O_\alpha \to Y$ given by $\sigma|_{O_\alpha} = \sigma_\alpha$.

We end this section with a comment on normalization. There is no loss in generality if we consider for all α, β and for all i

$$\varphi^i_{\alpha\alpha} = id \ , \ f^i_{\alpha\alpha\beta} = 1 \ , \ f^i_{\alpha\beta\beta} = 1 \tag{111}$$

Indeed first notice from (106) and (107) that $\varphi^i_{\alpha\alpha} = Ad_{f^i_{\alpha\alpha\alpha}}$ and $\varphi^i_{\alpha\alpha}(f^i_{\alpha\alpha\beta}) = f^i_{\alpha\alpha\alpha}$ so that $f^i_{\alpha\alpha\beta} = f^i_{\alpha\alpha\alpha}|_{O_{\alpha\beta}}$. Now, if $f^i_{\alpha\alpha\alpha} \neq 1$ consider the stably equivalent local data obtained from $\mathcal{E}'_{\alpha\beta} \equiv \eta_{\alpha\beta}\mathcal{E}_{\alpha\beta}$ where $\eta_{\alpha\beta} = \{\ell^i_{\alpha\beta}\ell^{j\,-1}_{\alpha\beta}, Ad_{\ell^i_{\alpha\beta}}\}$ with $\ell^i_{\alpha\beta} = f^{i\,-1}_{\alpha\alpha\alpha}|_{O_{\alpha\beta}}$. From (109) we have $\varphi'^i_{\alpha\alpha} = id$; from (110) we have $f'^i_{\alpha\alpha\beta} = 1$, it then also follows $f'^i_{\alpha\beta\beta} = 1$.

5. **Nonabelian Gerbes from Groups Extensions**

We here associate a bundle gerbe on the manifold M to every group extension

$$1 \to H \to E \xrightarrow{\pi} G \to 1 \tag{112}$$

and left principal G bundle P over M. We identify G with the coset ${}_H\backslash E$ so that E is a left H principal bundle. E is naturally a bibundle, the right action too is given by the group law in E

$$e \triangleleft h = eh = (ehe^{-1})e \tag{113}$$

thus $\varphi_e(h) = ehe^{-1}$. We denote by $\tau : P^{[2]} \to G,\ \tau(p_1, p_2) = g_{12}$ the map that associates to any two points p_1, p_2 of P that live on the same fiber the unique element $g_{12} \in G$ such that $p_1 = g_{12}p_2$. Let $\mathcal{E} \equiv \tau^*(E)$ be the pull-back of E on $P^{[2]}$, explicitly $\mathcal{E} = \{(p_1, p_2; e) \mid \pi(e) = \tau(p_1, p_2) = g_{12}\}$. Similarly $\mathcal{E}_{12} = \{(p_1, p_2, p_3; e) \mid \pi(e) = \tau(p_1, p_2) = g_{12}\}$, for brevity of notations we set $e_{12} \equiv (p_1, p_2, p_3; e)$. Similarly with $\mathcal{E}_{23}$ and $\mathcal{E}_{13}$, while e_{13}^{-1} is a symbolic notation for a point of $\mathcal{E}_{13}^{-1}$. Recalling (15) we have

$$(he)_{13}^{-1} = (ek)_{13}^{-1} = k^{-1}\, e_{13}^{-1}\ ,\ \ e_{13}^{-1} \triangleleft^{-1} h = k\, e_{13}^{-1} \tag{114}$$

where $k = e^{-1}he$. We now consider the point

$$\boldsymbol{f}^{-1}(p_1, p_2, p_3) \equiv [e_{12}, e'_{23}, (ee')_{13}^{-1}] \in \mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1} \tag{115}$$

where the square bracket denotes, as in (12), the equivalence class under the H action[2]. Expression (115) is well defined because $\pi(ee') = \pi(e)\pi(e') = g_{12}g_{23} = g_{13}$ the last equality following from $p_1 = g_{12}p_2\,,\ p_2 = g_{23}p_3\,,\ p_1 = g_{13}p_3\,$. Moreover $\boldsymbol{f}(p_1, p_2, p_3)$ is independent from e and e', indeed let $\hat{e}$ and $\hat{e}'$ be two other elements of E such that $\pi(\hat{e}) = \pi(e)\,,\ \pi(\hat{e}') = \pi(e')$; then $\hat{e} = he,\ \hat{e}' = h'e'$ with $h, h' \in H$ and $[\hat{e}_{12}, \hat{e}'_{23}, (\hat{e}\hat{e}')_{13}^{-1}] = [h\, e_{12}, h'\, e'_{23}, e'^{-1}h'^{-1}e^{-1}h^{-1}ee'\, (ee')_{13}^{-1}] = [e_{12}, e'_{23}, (ee')_{13}^{-1}]$. This shows that (115) defines a global section $\boldsymbol{f}^{-1}$ of $\mathcal{T} \equiv \mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1}$. Using the second relation in (114) we also have that $\boldsymbol{f}^{-1}$ is central so that $\mathcal{T}$ is a trivial bibundle. Finally (the inverse of) condition (92) is easily seen to hold and we conclude that $(\mathcal{E}, P, M, \boldsymbol{f})$ is a bundle gerbe. It is the so-called *lifting bundle gerbe*.

6. **Bundle Gerbes Modules**

The definition of a module for a nonabelian bundle gerbe is inspired by the abelian case [6].

Definition 16. *Given an H-bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$, an $\mathcal{E}$-module consists of a triple $(\mathcal{Q}, \mathcal{Z}, \boldsymbol{z})$ where $\mathcal{Q} \to Y$ is a D-H bundle, $\mathcal{Z} \to Y^{[2]}$ is a trivial D-bibundle and $\boldsymbol{z}$ is a global central section of $\mathcal{Z}^{-1}$ such that:*
i) on $Y^{[2]}$

$$\mathcal{Q}_1\mathcal{E} = \mathcal{Z}\mathcal{Q}_2 \tag{116}$$

and moreover

$$\boldsymbol{\varphi}_{12} = \boldsymbol{\psi}_1^{-1} \circ {}^{\bar{z}_{12}^{-1}} \circ \boldsymbol{\psi}_2. \tag{117}$$

ii) (116) is compatible with the bundle gerbe structure of $\mathcal{E}$, i.e. from (116) we have $\mathcal{Q}_1\mathcal{T} = \mathcal{Z}_{12}\mathcal{Z}_{23}\mathcal{Z}_{13}^{-1}\mathcal{Q}_1$ on $Y^{[3]}$ and we require that

$$\boldsymbol{z}_{23}\boldsymbol{z}_{12} = \boldsymbol{z}_{13}\boldsymbol{\xi}_1(\boldsymbol{f}) \tag{118}$$

holds true.

[2] It can be shown that a realization of the equivalence class $[e_{12}, e'_{23}] \in \mathcal{E}_{12}\mathcal{E}_{23}$ is given by $(p_1, p_2, p_3; ee')$ where ee' is just the product in E. (We won't use this property).

Remark 17. Let us note that the pair $(\mathcal{Z}, \boldsymbol{z}^{-1})$ and the pair $(\mathcal{T}, \boldsymbol{f}^{-1})$ in the above definition give the isomorphisms

$$z : \mathcal{Q}_1\mathcal{E} \to \mathcal{Q}_2 \quad , \quad f : \mathcal{E}_{12}\mathcal{E}_{23} \to \mathcal{E}_{13} \tag{119}$$

respectively of D-H bundles on $Y^{[2]}$ and of bibundles on $Y^{[3]}$. Condition *ii)* in Definition 16 is then equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{Q}_1\mathcal{E}_{12}\mathcal{E}_{23} & \xrightarrow{id_1 f} & \mathcal{Q}_1\mathcal{E}_{13} \\ {\scriptstyle z_{12}\, id_3}\big\downarrow & & \big\downarrow{\scriptstyle z_{13}} \\ \mathcal{Q}_2\mathcal{E}_{23} & \xrightarrow{z_{23}} & \mathcal{Q}_3 \end{array} \tag{120}$$

Definition 18. *We call two bundle gerbe modules $(\mathcal{Q}, \mathcal{Z}, \boldsymbol{z})$ and $(\mathcal{Q}', \mathcal{Z}', \boldsymbol{z}')$ (with the same crossed module structure) equivalent if:*
i) $\mathcal{Q}$ and $\mathcal{Q}'$ are isomorphic as D-H bundles; we write $\mathcal{Q} = \mathcal{I}\mathcal{Q}'$ where the D-bibundle $\mathcal{I}$ has global central section $\boldsymbol{i}^{-1}$ and $\boldsymbol{\psi} = {}^{\bar{i}^{-1}} \circ \boldsymbol{\psi}'$
ii) the global central sections $\boldsymbol{z}$, $\boldsymbol{z}'$ and $\boldsymbol{i}^{-1}$ satisfy the condition $\boldsymbol{z}'_{12} = \boldsymbol{i}_2^{-1}\boldsymbol{z}_{12}\boldsymbol{i}_1$.

Let us now assume that we have two stably equivalent bundle gerbes $(\mathcal{E}, Y, M, \boldsymbol{f})$ and $(\mathcal{E}', Y', M, \boldsymbol{f}')$ with $Y' = Y$. We have [cf. (93), (94)] $\eta_{12}\mathcal{E}_{12} = \mathcal{N}_1\mathcal{E}'_{12}\mathcal{N}_2^{-1}$ and $\boldsymbol{\varphi}_1(\boldsymbol{f}') = \boldsymbol{\ell}_{13}^{-1}\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}$. Let $\mathcal{Q}$ be an $\mathcal{E}$-module and $\mathcal{I}$ a trivial D-bibundle with a global central section $\boldsymbol{i}^{-1}$. It is trivial to check that $\mathcal{I}\mathcal{Q}\mathcal{N}$ is an $\mathcal{E}'$-module with $\mathcal{Z}'_{12} = \mathcal{I}_1\boldsymbol{\xi}_1(\eta_{12})\mathcal{Z}_{12}\mathcal{I}_2^{-1}$ and $\boldsymbol{z}'_{12} = \boldsymbol{i}_2^{-1}\boldsymbol{z}_{12}\boldsymbol{\xi}(\eta_{12})\boldsymbol{i}_1$. It is now easy to compare modules of stably equivalent gerbes that in general have $Y \neq Y'$.

Proposition 19. *Stably equivalent gerbes have the same equivalence classes of modules.*

Now we give the description of bundle gerbes modules in terms of local data on M. Let $\{EE_{\alpha\beta}\}$ be a gerbe in the sense of definition 15.

Definition 20. *A module for the gerbe $\{\mathcal{E}_{\alpha\beta}\}$ is given by a collection $\{\mathcal{Q}_\alpha\}$ of D-H bundles such that on double intersections $O_{\alpha\beta}$ there exist trivial D-bibundles $\mathcal{Z}_{\alpha\beta}$, $\mathcal{Q}_\alpha\mathcal{E}_{\alpha\beta} = \mathcal{Z}_{\alpha\beta}\mathcal{Q}_\beta$, with global central sections $\boldsymbol{z}_{\alpha\beta}$ of $\mathcal{Z}_{\alpha\beta}^{-1}$ such that on triple intersections $O_{\alpha\beta\gamma}$*

$$\boldsymbol{z}_{\beta\gamma}\boldsymbol{z}_{\alpha\beta} = \boldsymbol{z}_{\alpha\gamma}\boldsymbol{\xi}_\alpha(\boldsymbol{f}_{\alpha\beta\gamma}) \tag{121}$$

and on double intersections $O_{\alpha\beta}$

$$\boldsymbol{\varphi}_{\alpha\beta} = \boldsymbol{\psi}_\alpha^{-1} \circ {}^{\bar{z}_{\alpha\beta}^{-1}} \circ \boldsymbol{\psi}_\beta. \tag{122}$$

Canonical module.
For each H-bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ we have a canonical $\mathcal{E}$-module associated with it; it is constructed as follows. As a left $Aut(H)$-bundle the canonical module is simply the trivial bundle over Y. The right action of H is given by the canonical homomorphism $Ad : H \to Aut(H)$. For $(y, \eta) \in Y \times Aut(H)$ we have $\boldsymbol{\xi}_{(y,\eta)}(h) = \eta \circ Ad_h \circ \eta^{-1} = Ad_{\eta(h)}$and $\boldsymbol{\psi}_{(y,\eta)}(h) = \eta(h)$. The $Aut(H)$-H bundle morphism $z : (Y \times Aut(H))_1\mathcal{E} \to (Y \times Aut(H))_2$ is given in the following way. A generic element of $(Y \times Aut(H))_1\mathcal{E}$ is of the form $[(y, y', (y, \eta)), e]$ where $\eta \in Aut(H), (y, y') \in Y^{[2]}$and $e \in \mathcal{E}$ such that $p_1 \circ p(e) = y$ and $p_2 \circ p(e) = y'$. Here p is the projection $p : \mathcal{E} \to Y^{[2]}$. We set

$$z([(y, y', (y, \eta)), e]) = (y, y', (y', \eta \circ \boldsymbol{\varphi}_e)).$$

The commutativity of diagram (120) is equivalent to the following statement

$$\eta \circ \boldsymbol{\varphi}_{f[e_1,e_2]} = \eta \circ \boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}$$

and this is a consequence of the isomorphism of H-bibundles

$$f : \mathcal{E}_{12}\mathcal{E}_{23} \to \mathcal{E}_{13}.$$

We have

$$f([e_1, e_2]h) = (f[e_1, e_2])h\,,$$

but we also have

$$f([e_1, e_2)]h) = f(\boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}(h)[e_1, e_2]) = \boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}(h) f([e_1, e_2]).$$

On the other hand we can write

$$(f[e_1, e_2])h = \boldsymbol{\varphi}_{f[e_1,e_2]}(h) f[e_1, e_2].$$

Hence

$$\boldsymbol{\varphi}_{f[e_1,e_2]}(h) = \boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}(h)$$

and the commutativity of diagram (120) follows. We denote the canonical module as *can* in the following.

In the case of a bundle gerbe $\mathcal{E}$ associated with the lifting of a G-principal bundle P, as described in Section 5, we have another natural module. We follow the notation of Section 5. In the exact sequence of groups (112)

$$1 \to H \to E \xrightarrow{\pi} G \to 1\,,$$

H is a normal subgroup. This gives the group H the structure of a crossed E-module.

The $\mathcal{E}$-module $\mathcal{Q}$ is simply the trivial E-H bundle $P \times E \to P$. The D-H bundle morphims $z' : \mathcal{Q}_1\mathcal{E} \to \mathcal{Q}_2$ is given by (recall $p_1 = \pi(\tilde{e})p_2$)

$$z'[(p_1, p_2, (p_1, e), (p_1, p_2, \tilde{e})] = (p_1, p_2, (p_2, e\tilde{e})),$$

which of course is compatible with the bundle gerbe structure of $\mathcal{E}$. Due to the exact sequence (112) we do have a homomorphism $E \to Aut(\mathrm{H})$ and hence we have a map

$$t : Y \times E \to Y \times Aut(H),$$

which is a morphism between the modules compatible with the module structures, i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathcal{Q}_1\mathcal{E}_{12} & \xrightarrow{z_{12}} & \mathcal{Q}_2 \\ {\scriptstyle t}\Big\downarrow & & \Big\downarrow{\scriptstyle t} \\ can_1\mathcal{E}_{12} & \xrightarrow{z'_{12}} & can_2 \end{array} \tag{123}$$

More generally given any bundle gerbe $\mathcal{E}$ and an $\mathcal{E}$-module $\mathcal{Q}$ we have the trivial $Aut(H)$-H bundle $Aut(H) \times_D \mathcal{Q}$ (see Section 2). This gives a morphism $t : \mathcal{Q} \mapsto can$.

Now suppose that the bundle gerbe $\mathcal{E}$ is trivialized (stably equivalent to a trivial bundle gerbe) by $\mathcal{E}_{12} \sim \mathcal{N}_1^{-1}\mathcal{N}_2$ with $\mathcal{N}$ an H-bibundle on Y, hence a $\mathcal{E}$-module satisfies

$$\mathcal{Q}_2 \sim \mathcal{Q}_1\mathcal{E}_{12} \sim \mathcal{Q}_1\mathcal{N}_1^{-1}\mathcal{N}_2,$$

hence

$$\mathcal{Q}_2\mathcal{N}_2^{-1} \sim \mathcal{Q}_1\mathcal{N}_1^{-1} \tag{124}$$

It easily follows from (124) that $\mathcal{Q}\mathcal{N}^{-1} \to Y$ gives descent data for a D-H bundle $\widetilde{Q}$ over M. Conversely given a D-H bundle $\widetilde{Q} \to M$ the bundle $p^*(\widetilde{Q})\mathcal{N}$ is a $\mathcal{E}$-module. This proves the following

Proposition 21. *For a trivial bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ the category of $\mathcal{E}$-modules is equivalent to the category of D-H bundles over the base space M.*

7. Bundle Gerbe Connections

Definition 22. *A bundle gerbe connection on a bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ is a 2-connection $(\boldsymbol{a}, A)$ on $\mathcal{E} \to Y^{[2]}$ such that*

$$\boldsymbol{a}_{12} \dot{+} \boldsymbol{a}_{23} = f^*\boldsymbol{a}_{13}\,, \tag{125}$$

or which is the same

$$\boldsymbol{a}_{12} \dot{+} \boldsymbol{a}_{23} \dot{+} \boldsymbol{a}_{13}^r = \overline{f}^{-1} d\overline{f} + T_{A_1}(\overline{f}^{-1}) \tag{126}$$

holds true.

In the last equation $\overline{f}^{-1}$ is the bi-equivariant map $\overline{f}^{-1} : \mathcal{T} \to H$ associated with the global central section $\boldsymbol{f}^{-1}$ of $\mathcal{T}$. Moreover we used that $(\boldsymbol{a}^r, A^r)$ is a right 2-connection on $\mathcal{E}$ and a left 2-connection on $\mathcal{E}^{-1}$ [cf. (15)].

Remark 23. It follows from (125) that for a bundle gerbe connection $A_{12} = A_{13}$ must be satisfied, hence A is a pull-back via p_1 on $Y^{[2]}$ of a one form defined on Y. We can set $A_1 \equiv A_{12} = A_{13}$. Definition 21 contains implicitly the requirement that $(\boldsymbol{a}_{12}, \boldsymbol{a}_{23})$ are summable, which means that $A_1^r = A_2$. More explicitly (see (53)):

$$\mathcal{A}_1 + ad_{\boldsymbol{a}_{12}} = \varphi_{12}\mathcal{A}_2\varphi_{12}^{-1} + \varphi_{12}d\varphi_{12}^{-1}. \tag{127}$$

The affine sum of bundle gerbe connections is again a bundle gerbe connection. This is a consequence of the following affine property for sums of 2-connection. If on the bibundles E_1 and E_2 we have two couples of summable connections $(\boldsymbol{a}_1, \boldsymbol{a}_2)$, $(\boldsymbol{a}'_1, \boldsymbol{a}'_2)$, then $\lambda\boldsymbol{a}_1 + (1-\lambda)\boldsymbol{a}'_1$ is summable to $\lambda\boldsymbol{a}_2 + (1-\lambda)\boldsymbol{a}'_2$ and the sum is given by

$$(\lambda\boldsymbol{a}_1 + (1-\lambda)\boldsymbol{a}'_1) \dot{+} (\lambda\boldsymbol{a}_2 + (1-\lambda)\boldsymbol{a}'_2) = \lambda(\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2) + (1-\lambda)(\boldsymbol{a}'_1 \dot{+} \boldsymbol{a}'_2)\ . \tag{128}$$

We have the following theorem:

Theorem 24. *There exists a bundle gerbe connection $(\boldsymbol{a}, A)$ on each bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$.*

Proof. Let us assume for the moment the bundle gerbe to be trivial, $\mathcal{E} = \mathcal{N}_1^{-1}\mathcal{Z}\mathcal{N}_2$ with a bibundle $\mathcal{N} \to Y$ and a trivial bibundle $\mathcal{Z} \to Y^{[2]}$ with global central section $\boldsymbol{z}^{-1}$. Consider on $\mathcal{Z}$ the 2-connection $(\boldsymbol{\alpha}, \tilde{A})$, where the Lie$(Aut(H))$-valued one-form $\tilde{A}$ on Y is the pull-back of a one-form on M. Here $\boldsymbol{\alpha}$ is canonically determined by $\tilde{A}$ and $\boldsymbol{z}^{-1}$, we have $\boldsymbol{\alpha} = \bar{z}^{-1}d\bar{z} + T_A(\bar{z}^{-1})$. Next consider on $\mathcal{N}$ an arbitrary 2-connection $(\tilde{\boldsymbol{a}}, \tilde{A})$. Since $\tilde{A}$ is the pull-back of a one form on M we have that the sum $\boldsymbol{a} = \tilde{\boldsymbol{a}}_1^r \dot{+} \boldsymbol{\alpha} \dot{+} \tilde{\boldsymbol{a}}_2$ is well defined and that $(\boldsymbol{a}, A \equiv \tilde{A}^r)$ is a 2-connection on $\mathcal{E}$. Notice that under the canonical identification $\mathcal{Z}_{12}\mathcal{N}_2\mathcal{N}_2^{-1}\mathcal{Z}_{23} = \mathcal{Z}_{12}\mathcal{Z}_{23}$ we have the canonical identification $\boldsymbol{\alpha}_{12} \dot{+} \tilde{\boldsymbol{a}}_2 \dot{+} \tilde{\boldsymbol{a}}_2^r \dot{+} \boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_{12} \dot{+} \boldsymbol{\alpha}_{23}$. The point here is that $\mathcal{N}_2\mathcal{N}_2^{-1}$ has the canonical section

$\mathbf{1} = [n, n^{-1}]$, $n \in \mathcal{N}_2$, and that $\tilde{\boldsymbol{a}}_2 + \tilde{\boldsymbol{a}}_2^r = \bar{\mathbf{1}} d\bar{\mathbf{1}}^{-1} + T_A(\bar{\mathbf{1}})$ independently from $\tilde{\boldsymbol{a}}_2$. Then from $\mathcal{E} = \mathcal{N}_1^{-1}\mathcal{Z}\mathcal{N}_2$ we have $\mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1} = \mathcal{N}_1^{-1}\mathcal{Z}_{12}\mathcal{Z}_{23}\mathcal{Z}_{13}^{-1}\mathcal{N}_1$ and for the connections we have

$$\boldsymbol{a}_{12} + \boldsymbol{a}_{23} + \boldsymbol{a}_{13}^r = \tilde{\boldsymbol{a}}_1^r + \boldsymbol{\alpha}_{12} + \boldsymbol{\alpha}_{23} + \boldsymbol{\alpha}_{13}^r + \tilde{\boldsymbol{a}}_1 \ . \tag{129}$$

We want to prove that the r.h.s. of this equation equals the canonical 2-connection $\overline{f}^{-1} d\overline{f} + T_{A_1}(\overline{f}^{-1})$ associated with the trivial bundle $\mathcal{T}$ with section $\boldsymbol{f}^{-1}$. We first observe that a similar property holds for the sections of $\mathcal{Z}_{12}\mathcal{Z}_{23}\mathcal{Z}_{13}^{-1}$ and of $\mathcal{T}$: $\boldsymbol{f}^{-1} = \boldsymbol{\varphi}_1^{-1}(\boldsymbol{z}_{12}^{-1}\boldsymbol{z}_{23}^{-1}\boldsymbol{z}_{13}) \equiv \boldsymbol{n}_1^{-1}\boldsymbol{z}_{12}^{-1}\boldsymbol{z}_{23}^{-1}\boldsymbol{z}_{13}\boldsymbol{n}_1$ independently from the local section $\boldsymbol{n}_1$ of $\mathcal{N}_1$. Then one can explicitly check that this relation implies the relation $\tilde{\boldsymbol{a}}_1^r + \boldsymbol{\alpha}_{12} + \boldsymbol{\alpha}_{23} + \boldsymbol{\alpha}_{13}^r + \tilde{\boldsymbol{a}}_1 = \overline{f}^{-1} d\overline{f} + T_{A_1}(\overline{f}^{-1})$. This proves the validity of the theorem in the case of a trivial bundle gerbe. According to Theorem 14 any gerbe is locally trivial, so we can use the affine property of bundle gerbe connections and a partition of unity subordinate to the covering $\{O_\alpha\}$ of M to extend the proof to arbitrary bundle gerbes. □

A natural question arises: can we construct a connection on the bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ starting with:

- its nonabelian Čech cocycle $\overline{f}^{-1} : \mathcal{T} \to H$, $\boldsymbol{\varphi} : \mathcal{E} \times H \to H$
- sections $\sigma_\alpha : O_\alpha \to Y$
- a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\{O_\alpha\}$ of M?

The answer is positive. Let us describe the construction. First we use the local sections σ_α to map $Y|_{O_\alpha}^{[2]}$ to $Y^{[3]}$ via the map $r_\alpha^{[2]} : [y, y'] \mapsto [\sigma_\alpha(x), y, y']$, where $\wp(y) = \wp(y') = x$, similarly $r_\alpha^{[1]} : Y|_{O_\alpha}^{[1]} \to Y^{[2]}$. Next let us introduce the following H-valued one form $\boldsymbol{a}$

$$\boldsymbol{a} = \sum_\alpha \rho_\alpha r_\alpha^{[2]*} \boldsymbol{\varphi}_{12}^{-1} \left(\overline{f} \, d \, \overline{f}^{-1} \right) . \tag{130}$$

We easily find that

$$l^{h*}\boldsymbol{a} = Ad_h \boldsymbol{a} + p_1^* \left(h \sum_\alpha \rho_\alpha r_\alpha^{[1]*} \boldsymbol{\varphi}^{-1} (d \, \boldsymbol{\varphi}(h^{-1}) \right) .$$

The Lie$(Aut(H))$-valued 1-form $\sum_\alpha \rho_\alpha r_\alpha^{[1]*} \boldsymbol{\varphi}^{-1} \, d \, \boldsymbol{\varphi}$ is, due to (5), well defined on Y. We set

$$A = \sum_\alpha \rho_\alpha r_\alpha^{[1]*} \boldsymbol{\varphi}^{-1} \, d \, \boldsymbol{\varphi} - d \tag{131}$$

for the sought Lie$(Aut(H))$-valued 1-form on Y. Using the cocycle property of $\bar{f}$ and $\boldsymbol{\varphi}$ we easily have

Proposition 25. *Formulas (130) and (131) give a bundle gerbe connection.*

Using (88) we obtain that the 2-curvature $(\boldsymbol{k}, K)$ of the bundle gerbe 2-connection $(\boldsymbol{a}, A)$ satisfies

$$\boldsymbol{k}_{12} + \boldsymbol{k}_{23} + \boldsymbol{k}_{13}^r = T_{K_1}(\overline{f}^{-1}) \ . \tag{132}$$

Connection on a lifting bundle gerbe.
Let us now consider the example of a lifting bundle gerbe associated with an exact sequence of groups (112) and a G-principal bundle $P \to M$ on M. In this case, for any given connection $\bar{A}$ on P we can construct a connection on the lifting bundle gerbe. Let us choose a section $s\colon$ Lie(G) $\to$Lie(E); i.e a linear map such that $\pi \circ s = id$. We first define $A = s(\bar{A})$ and then consider the Lie(E) valued one-forms on $P^{[2]}$ given by $A_1 = p_1^* s(\bar{A})$ and $A_2 = p_2^* s(\bar{A})$, were p_1 and p_2 are respectively the projections onto the first and second factor of $P^{[2]}$. We next consider the one-form $\boldsymbol{a}$ on $\mathcal{E}$ that on $(p_1, p_2; e) \in \mathcal{E}$ is given by

$$\boldsymbol{a} \equiv e\mathcal{A}_2 e^{-1} + ede^{-1} - \mathcal{A}_1 \ , \tag{133}$$

here $\mathcal{A}_1 = p^*(A_1)$ and $\mathcal{A}_2 = p^*(A_2)$, with $p\,:\,\mathcal{E} \to P^{[2]}$. It is easy to see that $\pi^*\boldsymbol{a} = 0$ and that therefore $\boldsymbol{a}$ is Lie(H) valued; moreover $(\boldsymbol{a}, ad_A)$ is a 2-connection on $\mathcal{E}$. Recalling that on $\mathcal{E}$ we have $\boldsymbol{\varphi}_{(p_1,p_2;e)} = Ad_e$, it is now a straightforward check left to the reader to show that $(\boldsymbol{a}, ad_A)$ is a connection on the lifting bundle gerbe.

Connection on a module.
Let us start discussing the case of the canonical module $can = Aut(H) \times Y$ (see Section 6). Let $(\boldsymbol{a}, A)$ be a connection on our bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$. The Lie$(Aut(H))$-valued one-form A on Y lifts canonically to the connection $\tilde{\mathcal{A}}$ on can defined, forall $(\eta, y) \in can$, by $\tilde{\mathcal{A}} = \eta\mathcal{A}\eta^{-1} + \eta d\eta^{-1}$. Let us consider the following diagram

$$\begin{tikzcd} can_1 \oplus \mathcal{E} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_\oplus"] & \mathcal{E} & \\ can_1 & can_1\mathcal{E} \arrow[r, "z"] & can_2 \,. \end{tikzcd} \tag{134}$$

As in the case of the bundle gerbe connection we can consider whether the Lie$(Aut(H))$-valued one-form $\tilde{\mathcal{A}}_1 + \boldsymbol{\xi}(\boldsymbol{a})$ that lives on $can_1 \oplus \mathcal{E}$ is the pull-back under $\pi_\oplus$ of a one-form connection on $can_1\mathcal{E}$. If this is the case then we say that $\tilde{\mathcal{A}}_1$ and $\boldsymbol{a}$ are summable and we denote by $\tilde{\mathcal{A}}_1 \dot{+} ad_{\boldsymbol{a}}$ the resulting connection on $can_1\mathcal{E}$. Let us recall that on can we have $\boldsymbol{\xi}_{(\eta,y)} = Ad \circ \boldsymbol{\psi}_{(\eta,y)}$ with $\boldsymbol{\psi}_{(\eta,y)}(h) = \eta(h)$. It is now easy to check that $\tilde{\mathcal{A}}_1$ and $\boldsymbol{a}$ are summable and that their sum equals the pull-back under z of the connection $\tilde{\mathcal{A}}_2$; in formulae

$$\tilde{\mathcal{A}}_1 \dot{+} ad_{\boldsymbol{a}} = z^* \tilde{\mathcal{A}}_2 \,. \tag{135}$$

We also have that equality (135) is equivalent to the summability condition (127) for the bundle gerbe connection $\boldsymbol{a}$. Thus (135) is a new interpretation of the summability condition (127).

We now discuss connections on an arbitrary module $(\mathcal{Q}, \mathcal{Z}, \boldsymbol{z})$ associated with a bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ with connection $(\boldsymbol{a}, A)$. There are two natural requirements that a left connection $\mathcal{A}^D$ on the left D-bundle $\mathcal{Q}$ has to satisfy in order to be a module connection. The first one is that the induced connection $\widehat{\mathcal{A}^D}$ on $Aut(H) \times_D \mathcal{Q}$ has to be equal (under the isomorphism σ) to the connection $\tilde{\mathcal{A}}$ of can. This condition reads

$$\mathcal{A}^D = \boldsymbol{\psi}\mathcal{A}\boldsymbol{\psi}^{-1} + \boldsymbol{\psi} d\boldsymbol{\psi}^{-1} \ , \tag{136}$$

where in the l.h.s. $\mathcal{A}^D$ is tought to be Lie$(Aut(H))$ valued. In other words on Y we require $\sigma^*\widehat{\mathcal{A}^D} = A$, where σ is the global section of $Aut(H) \times_D \mathcal{Q}$.

Next consider the diagram

$$
\begin{tikzcd}
\mathcal{Q}_1 \oplus \mathcal{E} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[drr, "\pi_\oplus"] & \mathcal{E} & & \\
\mathcal{Q}_1 & & \mathcal{Q}_1\mathcal{E} \arrow[r, "z"] & \mathcal{Q}_2 \, .
\end{tikzcd}
\tag{137}
$$

We denote by $\mathcal{A}_1^D \dot{+} \alpha(\boldsymbol{a})$ the well defined D-connection on $\mathcal{Q}_1\mathcal{E}$ that pulled back on $\mathcal{Q}_1 \oplus \mathcal{E}$ equals $\pi_1^*\mathcal{A}_1^D + \boldsymbol{\xi}(\pi_2^*\boldsymbol{a})$. It is not difficult to see that $\mathcal{A}_1^D$ is indeed summable to $\boldsymbol{a}$ if forall $h \in H$, $\alpha(T_{\mathcal{A}^D}(h)) = \alpha(T_{\boldsymbol{\psi}\mathcal{A}\boldsymbol{\psi}^{-1}+\boldsymbol{\psi}d\boldsymbol{\psi}^{-1}}(h))$. This summability condition is thus implied by (136). The second requirement that $\mathcal{A}^D$ has to satify in order to be a module connection is

$$
\mathcal{A}_1^D \dot{+} \alpha(\boldsymbol{a}) = z^*\mathcal{A}_2^D \, . \tag{138}
$$

These conditions imply the summability condition (127) for the bundle gerbe connection $\boldsymbol{a}$.

Concerning the D-valued curvature $\mathcal{K}^D = d\mathcal{A}^D + \mathcal{A}^D \wedge \mathcal{A}^D$ we have

$$
\mathcal{K}_1^D \dot{+} \alpha(\boldsymbol{k_a}) = z_{12}^*\mathcal{K}_2^D \ . \tag{139}
$$

In terms of local data a gerbe connection consists of a collection of local 2-connections $(\boldsymbol{a}_{\alpha\beta}, A_\alpha)$ on the local bibundles $\mathcal{E}_{\alpha\beta} \to O_{\alpha\beta}$. For simplicity we assume the covering $\{O_\alpha\}$ to be a good one. The explicit relations that the local maps $f_{\alpha\beta\gamma} : O_{\alpha\beta\gamma} \to H$, $\varphi_{\alpha\beta} : O_{\alpha\beta} \to Aut(H)$ and the local representatives A_α, K_α, $a_{\alpha\beta}$ and $k_{\alpha\beta}$ (forms on O_α, $O_{\alpha\beta}$, etc.) satisfy are

$$
f_{\alpha\beta\gamma} f_{\alpha\gamma\delta} = \varphi_{\alpha\beta}(f_{\beta\gamma\delta}) f_{\alpha\beta\delta} \, , \tag{140}
$$

$$
\varphi_{\alpha\beta}\varphi_{\beta\gamma} = Ad_{f_{\alpha\beta\gamma}}\varphi_{\alpha\gamma} \, , \tag{141}
$$

$$
a_{\alpha\beta} + \varphi_{\alpha\beta}(a_{\beta\gamma}) = f_{\alpha\beta\gamma} a_{\alpha\gamma} f_{\alpha\beta\gamma}^{-1} + f_{\alpha\beta\gamma}\, d\, f_{\alpha\beta\gamma}^{-1} + T_{A_\alpha}(f_{\alpha\beta\gamma}) \, , \tag{142}
$$

$$
A_\alpha + ad_{a_{\alpha\beta}} = \varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} + \varphi_{\alpha\beta}\, d\, \varphi_{\alpha\beta}^{-1} \, , \tag{143}
$$

$$
k_{\alpha\beta} + \varphi_{\alpha\beta}(k_{\beta\gamma}) = f_{\alpha\beta\gamma} k_{\alpha\gamma} f_{\alpha\beta\gamma}^{-1} + T_{K_\alpha}(f_{\alpha\beta\gamma}) \tag{144}
$$

and

$$
K_\alpha + ad_{k_{\alpha\beta}} = \varphi_{\alpha\beta} K_\beta \varphi_{\alpha\beta}^{-1} \, . \tag{145}
$$

8. **Curving**

In this section we introduce the curving two form $\boldsymbol{b}$. This is achieved considering a gerbe stably equivalent to $(\mathcal{E}, Y, M, \boldsymbol{f})$. The resulting equivariant H-valued 3-form $\boldsymbol{h}$ is then shown to be given in terms of a form on Y. This description applies equally well to the abelian case; there one can however impose an extra condition [namely the vanishing of (147)]. We also give an explicit general construction of the curving $\boldsymbol{b}$ in terms of a partition of unity. This construction depends only on the partition of unity, and in the abelian case it naturally reduces to the usual one that automatically encodes the vanishing of (147).

Consider a bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ with connection $(\boldsymbol{a}, A)$ and curvature $(\boldsymbol{k_a}, K_A)$ and an H-bibundle $\mathcal{N} \to Y$ with a 2-connection $(\boldsymbol{c}, A)$. Then we have a stably equivalent gerbe $(\mathcal{N}_1^{-1}\mathcal{E}\mathcal{N}_2, Y, M, \boldsymbol{\varphi}_1^{-1}(\boldsymbol{f}))$ with connection $(\boldsymbol{\theta}, A^{r_1})$ given by

$$\boldsymbol{\theta} = \boldsymbol{c}_1^{r_1} \dot{+} \boldsymbol{a} \dot{+} \boldsymbol{c}_2 \,. \tag{146}$$

Also we can consider a K_A-equivariant horizontal 2-form $\boldsymbol{b}$ on $\mathcal{N}$. Again on the bibundle $\mathcal{N}_1^{-1}\mathcal{E}\mathcal{N}_2 \to Y^{[2]}$ we get a well defined $K_A^{r_1}$-equivariant horizontal 2-form

$$\tilde{\boldsymbol{\delta}} = \boldsymbol{b}_1^{r_1} \dot{+} \boldsymbol{k_a} \dot{+} \boldsymbol{b}_2 \,. \tag{147}$$

Contrary to the abelian case we cannot achieve $\tilde{\boldsymbol{\delta}} = 0$, unless K_A is inner (remember $\tilde{\boldsymbol{\delta}}$ is always $K_A^{r_1}$-equivariant). Next we consider the equivariant horizontal H-valued 3-form $\boldsymbol{h}$ on $\mathcal{N}$ given by

$$\boldsymbol{h} = D_{\boldsymbol{c}}\boldsymbol{b} \,. \tag{148}$$

Because of the Bianchi identity $dK_A + [A, K_A] = 0$ this is indeed an equivariant form on $\mathcal{N}$. Obviously the horizontal form $\boldsymbol{\varphi}^{-1}(\boldsymbol{h})$ is invariant under the left H-action

$$l^{h^*}\boldsymbol{\varphi}^{-1}(\boldsymbol{h}) = \boldsymbol{\varphi}^{-1}(\boldsymbol{h}) \tag{149}$$

and therefore it projects to a well defined form on Y.

Using now the property of the covariant derivative (86) and the Bianchi identity (68) we can write

$$\boldsymbol{h}_1^{r} \dot{+} \boldsymbol{h}_2 = D_{\boldsymbol{\theta}}\tilde{\boldsymbol{\delta}} \,. \tag{150}$$

Finally from (72) we get the Bianchi identity for $\boldsymbol{h}$

$$D_{\boldsymbol{c}}\boldsymbol{h} = [\boldsymbol{k_c}, \boldsymbol{b}] + T_{K_A}(\boldsymbol{k_c}) - T_{K_A}(\boldsymbol{b}) \,. \tag{151}$$

For the rest of this section we consider the special case where $\mathcal{N}$ is a trivial bibundle with global central section $\bar{\sigma}$ and with 2-connection given by $(\boldsymbol{c}, A)$, where $\boldsymbol{c}$ is canonically given by $\bar{\sigma}$,

$$\boldsymbol{c} = \bar{\sigma} d\bar{\sigma}^{-1} + T_A(\bar{\sigma}) \ .$$

Since the only H-bibundle $\mathcal{N} \to Y$ that we can canonically associate to a generic bundle gerbe is the trivial one (see Proposition 7), the special case where $\mathcal{N}$ is trivial seems quite a natural case.

In terms of local data curving is a collection $\{\boldsymbol{b}_\alpha\}$ of K_α-equivariant horizontal two forms on trivial H-bibundles $O_\alpha \times H \to O_\alpha$. Again we assume the covering O_α to be a good one and write out explicitly the relations to which the local representatives of b_α and h_α (forms on O_α) are subject:

$$k_{\alpha\beta} + \varphi_{\alpha\beta}(b_\beta) = b_\alpha + \delta_{\alpha\beta} \,, \tag{152}$$
$$\delta_{\alpha\beta} + \varphi_{\alpha\beta}(\delta_{\beta\gamma}) = f_{\alpha\beta\gamma}\delta_{\alpha\gamma}f_{\alpha\beta\gamma}^{-1} + T_{\nu_\alpha}(f_{\alpha\beta\gamma}) \,, \tag{153}$$
$$\nu_\alpha \equiv K_\alpha - ad_{b_\alpha} \,, \tag{154}$$
$$h_\alpha = d\, b_\alpha - T_{A_\alpha}(b_\alpha) \,, \tag{155}$$
$$\varphi_{\alpha\beta}(h_\beta) = h_\alpha + d\delta_{\alpha\beta} + [a_{\alpha\beta}, \delta_{\alpha\beta}] + T_{K_\alpha}(a_{\alpha\beta}) - T_{A_\alpha}(\delta_{\alpha\beta}) \tag{156}$$

and the Bianchi identity

$$d\, h_\alpha + T_{K_A}(b_\alpha) = 0 \,. \tag{157}$$

Here we introduced $\delta_{\alpha\beta} = \varphi_\alpha(\tilde{\delta}_{\alpha\beta})$. Equations (140)-(145) and (152)-(157) are the same as those listed after Theorem 10.1 in [26].

We now consider the case $Y = \sqcup O_\alpha$; this up to stable equivalence is always doable. Given a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\{O_\alpha\}$ of M, we have a natural

choice for the H-valued curving 2-form $\boldsymbol{b}$ on $\sqcup O_\alpha \times H$. It is the pull-back under the projection $\sqcup O_\alpha \times H \to \sqcup O_\alpha$ of the 2-form

$$\sqcup \sum_\beta \rho_\beta k_{\alpha\beta} \tag{158}$$

on $Y = \sqcup O_\alpha$. In this case we have for the local H-valued 2-forms $\delta_{\alpha\beta}$ the following expression

$$\begin{aligned} \delta_{\alpha\beta} &= \sum_\gamma \rho_\gamma (f_{\alpha\beta\gamma} k_{\alpha\gamma} f^{-1}_{\alpha\beta\gamma} - k_{\alpha\gamma} + T_{K_\alpha}(f_{\alpha\beta\gamma})) \\ &= \sum_\gamma \rho_\gamma (k_{\alpha\beta} + \varphi_{\alpha\beta}(k_{\beta\gamma}) - k_{\alpha\gamma})\,. \end{aligned} \tag{159}$$

We can now use Proposition 25 together with (158) in order to explicitly construct from the Čech cocycle $(\boldsymbol{f}, \boldsymbol{\varphi})$ an H-valued 3-form $\boldsymbol{h}$.

We conclude this final section by grouping together the global cocycle formulae that imply all the local expressions (140)-(145) and (152)-(157),

$$\boldsymbol{f}_{124}\, \boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234}) = \boldsymbol{f}_{134}\, \boldsymbol{f}_{123}\,, \tag{92}$$

$$\boldsymbol{a}_{12} + \boldsymbol{a}_{23} = f^* \boldsymbol{a}_{13}\,, \tag{125}$$

$$\tilde{\boldsymbol{\delta}} = \boldsymbol{b}_1^{r_1} + \boldsymbol{k}_{\boldsymbol{a}} + \boldsymbol{b}_2\,, \tag{147}$$

$$\boldsymbol{h} = D_{\boldsymbol{c}} \boldsymbol{b}\ . \tag{148}$$

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